

HYPERBOLICITY OF DIRECT PRODUCTS OF GRAPHS

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ABSTRACT. If X is a geodesic metric space and $x_1, x_2, x_3 \in X$, a *geodesic triangle* $T = \{x_1, x_2, x_3\}$ is the union of the three geodesics $[x_1x_2]$, $[x_2x_3]$ and $[x_3x_1]$ in X . The space X is δ -*hyperbolic* (in the Gromov sense) if any side of T is contained in a δ -neighborhood of the union of the two other sides, for every geodesic triangle T in X . If X is hyperbolic, we denote by $\delta(X)$ the sharp hyperbolicity constant of X , i.e., $\delta(X) = \inf\{\delta \geq 0 : X \text{ is } \delta\text{-hyperbolic}\}$. Some previous works characterize the hyperbolic product graphs (for the Cartesian, strong, join, corona and lexicographic products) in terms of properties of the factor graphs. However, the problem with the direct product is more complicated. In this paper, we prove that if the direct product $G_1 \times G_2$ is hyperbolic, then one factor is hyperbolic and the other one is bounded. Also, we prove that this necessary condition is, in fact, a characterization in many cases. In other cases, we find characterizations which are not so simple. Furthermore, we obtain formulae or good bounds for the hyperbolicity constant of the direct product of some important graphs.

Keywords: Direct product of graphs; Geodesics; Gromov hyperbolicity.

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1. INTRODUCTION

The different kinds of products of graphs are an important research topic. Some large graphs are composed from some existing smaller ones by using several products of graphs, and many properties of such large graphs are strongly associated with that of the corresponding smaller ones. In particular, given two graphs G_1, G_2 , the *direct product* $G_1 \times G_2$ is the graph with the vertex set $V(G_1 \times G_2)$, and such that two vertices (u_1, v_1) and (u_2, v_2) of $G_1 \times G_2$ are adjacent if $[u_1, u_2] \in E(G_1)$ and $[v_1, v_2] \in E(G_2)$. The direct product is clearly commutative and associative. Weichsel observed that $G_1 \times G_2$ is connected if and only if G_1 and G_2 are connected and G_1 or G_2 is not a bipartite graph [53]. Many different properties of direct product of graphs have been studied (sometimes with various different names, such as cardinal product, tensor product, Kronecker product, categorical product, conjunction,...). The study includes structural results [6, 13, 28, 31, 32, 33], hamiltonian properties [5, 36], and above all the well-known Hedetniemi's conjecture on chromatic number of direct product of two graphs (see [30] and [55]). Open problems in the area suggest that a deeper structural understanding of this product would be welcome.

Hyperbolic spaces play an important role in geometric group theory and in the geometry of negatively curved spaces (see [3, 25, 27]). The concept of Gromov hyperbolicity grasps the essence of negatively curved spaces like the classical hyperbolic space, simply connected Riemannian manifolds of negative sectional curvature bounded away from 0, and of discrete spaces like trees and the Cayley graphs of many finitely generated groups. It is remarkable that a simple concept leads to such a rich general theory (see [3, 25, 27]).

The first works on Gromov hyperbolic spaces deal with finitely generated groups (see [27]). Initially, Gromov spaces were applied to the study of automatic groups in the science of computation (see, e.g., [43]); indeed, hyperbolic groups are strongly geodesically automatic, i.e., there is an automatic structure on the

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group [18]. The concept of hyperbolicity appears also in discrete mathematics, algorithms and networking. For example, it has been shown empirically in [50] that the internet topology embeds with better accuracy into a hyperbolic space than into a Euclidean space of comparable dimension (formal proofs that the distortion is related to the hyperbolicity can be found in [52]); furthermore, it is evidenced that many real networks are hyperbolic (see, e.g., [1, 2, 21, 38, 42]). Another important application of these spaces is the study of the spread of viruses through the internet (see [34, 35]). Furthermore, hyperbolic spaces are useful in secure transmission of information on the network (see [34, 35]); also to traffic flow and effective resistance of networks [20, 26, 39]. The hyperbolicity has also been used extensively in the context of random graphs (see, e.g., [47, 48, 49]).

In [51] it was proved the equivalence of the hyperbolicity of many negatively curved surfaces and the hyperbolicity of a graph related to it; hence, it is useful to know hyperbolicity criteria for graphs from a geometrical viewpoint. Hence, the study of Gromov hyperbolic graphs is a subject of increasing interest; see, e.g., [1, 2, 8, 9, 14, 21, 34, 35, 40, 41, 45, 47, 48, 49, 51, 52, 54] and the references therein.

We say that a curve $\gamma : [a, b] \rightarrow X$ in a metric space X is a *geodesic* if we have $L(\gamma|_{[t,s]}) = d(\gamma(t), \gamma(s)) = |t-s|$ for every $s, t \in [a, b]$, where L and d denote length and distance, respectively, and $\gamma|_{[t,s]}$ is the restriction of the curve γ to the interval $[t, s]$ (then γ is equipped with an arc-length parametrization). The metric space X is said *geodesic* if for every couple of points in X there exists a geodesic joining them; we denote by $[xy]$ any geodesic joining x and y ; this notation is ambiguous, since in general we do not have uniqueness of geodesics, but it is very convenient. Consequently, any geodesic metric space is connected. If the metric space X is a graph, then the edge joining the vertices u and v will be denoted by $[u, v]$.

In order to consider a graph G as a geodesic metric space, identify (by an isometry) any edge $[u, v] \in E(G)$ with the interval $[0, 1]$ in the real line; then the edge $[u, v]$ (considered as a graph with just one edge) is isometric to the interval $[0, 1]$. Thus, the points in G are the vertices and, also, the points in the interior of any edge of G . In this way, any connected graph G has a natural distance defined on its points, induced by taking shortest paths in G , and we can see G as a metric graph. If x, y are in different connected components of G , we define $d_G(x, y) = \infty$.

Throughout this paper, $G = (V, E) = (V(G), E(G))$ denotes a connected simple (without loops and multiple edges) graph such that every edge has length 1 and $V \neq \emptyset$. These properties guarantee that G is a geodesic metric space. Note that to exclude multiple edges and loops is not an important loss of generality, since [9, Theorems 8 and 10] reduce the problem of computing the hyperbolicity constant of graphs with multiple edges and/or loops to the study of simple graphs.

If X is a geodesic metric space and $x_1, x_2, x_3 \in X$, the union of three geodesics $[x_1x_2]$, $[x_2x_3]$ and $[x_3x_1]$ is a *geodesic triangle* that will be denoted by $T = \{x_1, x_2, x_3\}$ and we will say that x_1, x_2 and x_3 are the vertices of T ; it is usual to write also $T = \{[x_1x_2], [x_2x_3], [x_3x_1]\}$. We say that T is δ -thin if any side of T is contained in the δ -neighborhood of the union of the two other sides. We denote by $\delta(T)$ the sharp thin constant of T , i.e., $\delta(T) := \inf\{\delta \geq 0 : T \text{ is } \delta\text{-thin}\}$. The space X is δ -hyperbolic (or satisfies the *Rips condition* with constant δ) if every geodesic triangle in X is δ -thin. We denote by $\delta(X)$ the sharp hyperbolicity constant of X , i.e., $\delta(X) := \sup\{\delta(T) : T \text{ is a geodesic triangle in } X\}$. We say that X is *hyperbolic* if X is δ -hyperbolic for some $\delta \geq 0$; then X is hyperbolic if and only if $\delta(X) < \infty$. If we have a triangle with two identical vertices, we call it a “bigon”. Obviously, every bigon in a δ -hyperbolic space is δ -thin. If X has connected components $\{X_i\}_{i \in I}$, then we define $\delta(X) := \sup_{i \in I} \delta(X_i)$, and we say that X is hyperbolic if $\delta(X) < \infty$.

In the classical references on this subject (see, e.g., [3, 12, 25]) appear several different definitions of Gromov hyperbolicity, which are equivalent in the sense that if X is δ -hyperbolic with respect to one definition, then it is δ' -hyperbolic with respect to another definition (for some δ' related to δ). The definition that we have chosen has a deep geometric meaning (see, e.g., [25]).

We want to remark that the main examples of hyperbolic graphs are the trees. In fact, the hyperbolicity constant of a geodesic metric space can be viewed as a measure of how “tree-like” the space is, since those spaces X with $\delta(X) = 0$ are precisely the metric trees. This is an interesting subject since, in many

applications, one finds that the borderline between tractable and intractable cases may be the tree-like degree of the structure to be dealt with (see, e.g., [19]).

For a finite graph with n vertices it is possible to compute $\delta(G)$ in time $O(n^{3.69})$ [24] (this is improved in [21, 23]). Given a Cayley graph (of a presentation with solvable word problem) there is an algorithm which allows to decide if it is hyperbolic [44]. However, deciding whether or not a general infinite graph is hyperbolic is usually very difficult. Thus, a way to approach the problem is to study hyperbolicity for particular types of graphs. In this line, many researches have studied the hyperbolicity of several classes of graphs: chordal graphs [7, 14, 40, 54], vertex-symmetric graphs [15], bipartite and intersection graphs [22], bridged graphs [37], expanders [39] and some products of graphs: Cartesian product [41], strong product [16], corona and join product [17].

In this paper we characterize in many cases the hyperbolic direct product of graphs. Here the situation is more complex than with the Cartesian or the strong product, which is in part due to the facts that the direct product of two bipartite graphs is already disconnected and that the formula for the distance in $G_1 \times G_2$ is more complicated than in the case of other products of graphs. Theorem 2.25 proves that if $G_1 \times G_2$ is hyperbolic, then one factor is hyperbolic and the other one is bounded. Also, we prove that this necessary condition is, in fact, a characterization in many cases. If G_1 is a hyperbolic graph and G_2 is a bounded graph, then we prove that $G_1 \times G_2$ is hyperbolic when G_2 has some odd cycle (Theorem 2.11) or G_1 and G_2 do not have odd cycles (Theorem 2.12). Otherwise, the characterization is a more difficult task; if G_1 has some odd cycle and G_2 does not have odd cycles, Theorems 2.26 and 2.28 provide sufficient conditions for non-hyperbolicity and hyperbolicity, respectively; besides, Theorems 2.37 and Corollary 2.38 characterize the hyperbolicity of $G_1 \times G_2$ under some additional conditions. Furthermore, we obtain formulae or good bounds for the hyperbolicity constant of the direct product of some important graphs (in particular, Theorem 3.6 provides the precise value of the hyperbolicity constant for many direct products of bipartite graphs).

2. HYPERBOLIC DIRECT PRODUCTS

In order to study the hyperbolicity constant of the direct product of two graphs $G_1 \times G_2$, we will need bounds for the distance between two arbitrary points. We will use the definition given in [29].

Definition 2.1. Let $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$ be two graphs. The direct product $G_1 \times G_2$ of G_1 and G_2 has $V(G_1) \times V(G_2)$ as vertex set, so that two distinct vertices (u_1, v_1) and (u_2, v_2) of $G_1 \times G_2$ are adjacent if $[u_1, u_2] \in E(G_1)$ and $[v_1, v_2] \in E(G_2)$.

If G_1 and G_2 are isomorphic, we write $G_1 \simeq G_2$. It is clear that if $G_1 \simeq G_2$, then $\delta(G_1) = \delta(G_2)$.

From the definition, it follows that the direct product of two graphs is commutative, i.e., $G_1 \times G_2 \simeq G_2 \times G_1$. Hence, the conclusion of every result in this paper with some “non-symmetric” hypothesis also holds if we change the roles of G_1 and G_2 (see, e.g., Theorems 2.11, 2.12, 2.26, 2.28 and 2.37 and Corollary 2.38).

In what follows we denote by π_i the projection $\pi_i : V(G_1 \times G_2) \rightarrow V(G_i)$ for $i \in \{1, 2\}$. Note that, in fact, this projection is well defined as a map $\pi_i : G_1 \times G_2 \rightarrow G_i$ for $i \in \{1, 2\}$.

We collect some previous results of [29], which will be useful. If G is a graph and $u, u' \in V(G)$, then by a u, u' -walk in G we mean a path joining u and u' where repeating vertices is allowed.

Proposition 2.2. [29, Proposition 5.7] Suppose (u, v) and (u', v') are vertices of the direct product $G_1 \times G_2$, and n is an integer for which G_1 has a u, u' -walk of length n and G_2 has a v, v' -walk of length n . Then $G_1 \times G_2$ has a walk of length n from (u, v) to (u', v') . The smallest such n (if it exists) equals $d_{G_1 \times G_2}((u, v), (u', v'))$. If no such n exists, then $d_{G_1 \times G_2}((u, v), (u', v')) = \infty$.

Proposition 2.3. [29, Proposition 5.8] Suppose x and y are vertices of $G_1 \times G_2$. Then

$$d_{G_1 \times G_2}(x, y) = \min \{n \in \mathbb{N} \mid \text{each factor } G_i \text{ has a } \pi_i(x), \pi_i(y)\text{-walk of length } n \text{ for } i = 1, 2\},$$

where it is understood that $d_{G_1 \times G_2}(x, y) = \infty$ if no such n exists.

Definition 2.4. The diameter of the vertices of the graph G , denoted by $\text{diam } V(G)$, is defined as

$$\text{diam } V(G) := \sup\{d_G(u, v) : u, v \in V(G)\},$$

and the diameter of the graph G , denoted by $\text{diam } G$, is defined as

$$\text{diam } G := \sup\{d_G(x, y) : x, y \in G\}.$$

Corollary 2.5. We have for every $(u, v), (u', v') \in V(G_1 \times G_2)$

$$d_{G_1 \times G_2}((u, v), (u', v')) \geq \max\{d_{G_1}(u, u'), d_{G_2}(v, v')\}$$

and, consequently,

$$\text{diam } V(G_1 \times G_2) \geq \max\{\text{diam } V(G_1), \text{diam } V(G_2)\}.$$

Furthermore, if $d_{G_1}(u, u')$ and $d_{G_2}(v, v')$ have the same parity, then

$$d_{G_1 \times G_2}((u, v), (u', v')) = \max\{d_{G_1}(u, u'), d_{G_2}(v, v')\}$$

and, consequently,

$$\text{diam } V(G_1 \times G_2) = \max\{\text{diam } V(G_1), \text{diam } V(G_2)\}.$$

In this paper by trivial graph we mean a graph having just a single vertex.

The following theorem, first proved by Weichsel in 1962, characterizes connectedness in direct products of two factors. As usual, by *cycle* we mean a simple closed curve, i.e., a path with different vertices, unless the last one, which is equal to the first vertex.

Theorem 2.6. [29, Theorem 5.9] Suppose G_1 and G_2 are connected non-trivial graphs. If at least one of G_1 or G_2 has an odd cycle, then $G_1 \times G_2$ is connected. If both G_1 and G_2 are bipartite, then $G_1 \times G_2$ has exactly two connected components.

Corollary 2.7. [29, Corollary 5.10] A direct product of connected non-trivial graphs is connected if and only if at most one of the factors is bipartite. In fact, the product has $2^{\max\{k, 1\}-1}$ connected components, where k is the number of bipartite factors.

Proposition 2.8. Let G_1 and G_2 be two unbounded graphs. Then $G_1 \times G_2$ is not hyperbolic.

Proof. Since G_1 and G_2 are unbounded graphs, for each positive integer n there exist two geodesic paths $P_1 := [w_1, w_2] \cup [w_2, w_3] \cup \dots \cup [w_{n-1}, w_n]$ in G_1 and $P_2 := [v_1, v_2] \cup [v_2, v_3] \cup \dots \cup [v_{n-1}, v_n]$ in G_2 . If n is odd, then we can consider the geodesic triangle T in $G_1 \times G_2$ defined by the following geodesics:

$$\gamma_1 := [(w_1, v_2), (w_2, v_1)] \cup [(w_2, v_1), (w_3, v_2)] \cup [(w_3, v_2), (w_4, v_1)] \cup \dots \cup [(w_{n-1}, v_1), (w_n, v_2)],$$

$$\gamma_2 := [(w_1, v_2), (w_2, v_3)] \cup [(w_2, v_3), (w_1, v_4)] \cup [(w_1, v_4), (w_2, v_5)] \cup \dots \cup [(w_1, v_{n-1}), (w_2, v_n)],$$

$$\gamma_3 := [(w_2, v_n), (w_3, v_{n-1})] \cup [(w_3, v_{n-1}), (w_4, v_{n-2})] \cup [(w_4, v_{n-2}), (w_5, v_{n-3})] \cup \dots \cup [(w_{n-1}, v_3), (w_n, v_2)],$$

Corollary 2.5 gives that $\gamma_1, \gamma_2, \gamma_3$ are geodesics.

Let $m := \frac{n+1}{2}$ and consider the vertex (w_m, v_{m+1}) in γ_3 . For every vertex (w_i, v_j) in γ_1 , $j \in \{1, 2\}$, we have $d_{G_1 \times G_2}((w_m, v_{m+1}), (w_i, v_j)) \geq d_{G_2}(v_{m+1}, v_j) \geq m+1-2 = \frac{n-1}{2}$ by Corollary 2.5. We have for every vertex (w_i, v_j) in γ_2 , $i \in \{1, 2\}$, by Corollary 2.5, $d_{G_1 \times G_2}((w_m, v_{m+1}), (w_i, v_j)) \geq d_{G_1}(w_m, w_i) \geq m-2 = \frac{n-3}{2}$. Hence, $d_{G_1 \times G_2}((w_m, v_{m+1}), \gamma_1 \cup \gamma_2) \geq \frac{n-3}{2}$ and $\delta(G_1 \times G_2) \geq \delta(T) \geq \frac{n-3}{2}$. Since n is arbitrarily large, $G_1 \times G_2$ is not hyperbolic. \square

Let (X, d_X) and (Y, d_Y) be two metric spaces. A map $f : X \rightarrow Y$ is said to be an (α, β) -quasi-isometric embedding, with constants $\alpha \geq 1$, $\beta \geq 0$ if, for every $x, y \in X$:

$$\alpha^{-1}d_X(x, y) - \beta \leq d_Y(f(x), f(y)) \leq \alpha d_X(x, y) + \beta.$$

The function f is ε -full if for each $y \in Y$ there exists $x \in X$ with $d_Y(f(x), y) \leq \varepsilon$.

A map $f : X \rightarrow Y$ is said to be a *quasi-isometry*, if there exist constants $\alpha \geq 1$, $\beta, \varepsilon \geq 0$ such that f is an ε -full (α, β) -quasi-isometric embedding.

Two metric spaces X and Y are *quasi-isometric* if there exists a quasi-isometry $f : X \rightarrow Y$. One can check that to be quasi-isometric is an equivalence relation. An (α, β) -*quasi-geodesic* in X is an (α, β) -quasi-isometric embedding between an interval of \mathbb{R} and X .

A fundamental property of hyperbolic spaces is the following (see, e.g., [25, p.88]):

Theorem 2.9 (Invariance of hyperbolicity). *Let $f : X \rightarrow Y$ be an (α, β) -quasi-isometric embedding between the geodesic metric spaces X and Y . If Y is hyperbolic, then X is hyperbolic.*

Besides, if f is ε -full for some $\varepsilon \geq 0$ (a quasi-isometry), then X is hyperbolic if and only if Y is hyperbolic.

Lemma 2.10. *Consider two graphs G_1 and G_2 . If $f : V(G_1) \rightarrow V(G_2)$ is an (α, β) -quasi-isometric embedding, then there exists an $(\alpha, \alpha + \beta)$ -quasi-isometric embedding $g : G_1 \rightarrow G_2$ with $g = f$ on $V(G_1)$. Furthermore, if f is ε -full, then g is $(\varepsilon + \frac{1}{2})$ -full.*

Proof. For each $x \in G_1$, let us choose a closest point $v_x \in V(G_1)$ from x , and define $g(x) := f(v_x)$. Note that $v_x = x$ if $x \in V(G_1)$ and so $g = f$ on $V(G_1)$. Given $x, y \in G_1$, we have

$$\begin{aligned} d_{G_2}(g(x), g(y)) &= d_{G_2}(f(v_x), f(v_y)) \leq \alpha d_{G_1}(v_x, v_y) + \beta \leq \alpha(d_{G_1}(x, y) + 1) + \beta, \\ d_{G_2}(g(x), g(y)) &= d_{G_2}(f(v_x), f(v_y)) \geq \alpha^{-1}d_{G_1}(v_x, v_y) - \beta \geq \alpha^{-1}(d_{G_1}(x, y) - 1) - \beta, \end{aligned}$$

and g is an $(\alpha, \alpha + \beta)$ -quasi-isometric embedding, since $\alpha \geq 1 \geq \alpha^{-1}$.

Furthermore, if f is ε -full, then g is $(\varepsilon + \frac{1}{2})$ -full since $g(G_1) = f(V(G_1))$. \square

Given a graph G , let $g_I(G)$ denote the *odd girth* of G , this is, the length of the shortest odd cycle in G .

Theorem 2.11. *Let G_1 be a graph and G_2 be a non-trivial bounded graph with some odd cycle. Then, $G_1 \times G_2$ is hyperbolic if and only if G_1 is hyperbolic.*

Proof. Let $v_0 \in V(G_2)$ such that v_0 is contained in an odd cycle C with $L(C) = g_I(G_2)$. Consider the map $i : V(G_1) \rightarrow V(G_1 \times G_2)$ such that $i(w) := (w, v_0)$ for every $w \in V(G_1)$.

By Corollary 2.5, for any pair of vertices $w_1, w_2 \in V(G_1)$, $d_{G_1}(w_1, w_2) \leq d_{G_1 \times G_2}((w_1, v_0), (w_2, v_0))$. Also, Proposition 2.3 gives the following.

If a geodesic joining w_1 and w_2 has even length, then

$$d_{G_1 \times G_2}((w_1, v_0), (w_2, v_0)) = d_{G_1}(w_1, w_2).$$

If a geodesic joining w_1 and w_2 has odd length, then C defines a v_0, v_0 -walk with odd length and

$$d_{G_1 \times G_2}((w_1, v_0), (w_2, v_0)) \leq \max\{d_{G_1}(w_1, w_2), g_I(G_2)\} \leq d_{G_1}(w_1, w_2) + g_I(G_2).$$

Thus, i is a $(1, g_I(G_2))$ quasi-isometric embedding.

Consider any $(w, v) \in V(G_1 \times G_2)$. Then, if the geodesic joining v and v_0 has even length,

$$d_{G_1 \times G_2}((w, v), (w, v_0)) = d_{G_2}(v, v_0).$$

If a geodesic joining v and v_0 has odd length, $[vv_0] \cup C$ defines a v, v_0 -walk with even length. Therefore,

$$d_{G_1 \times G_2}((w, v), (w, v_0)) \leq d_{G_2}(v, v_0) + g_I(G_2).$$

Thus, i is $(\text{diam}(V(G_2)) + g_I(G_2))$ -full.

Hence, by Lemma 2.10, there is a $(\text{diam}(V(G_2)) + g_I(G_2) + \frac{1}{2})$ -full $(1, g_I(G_2) + 1)$ -quasi-isometry, $j : G_1 \rightarrow G_1 \times G_2$, and $G_1 \times G_2$ is hyperbolic if and only if G_1 is hyperbolic by Theorem 2.9. \square

Theorem 2.12. *Let G_1 be a graph without odd cycles and G_2 be a non-trivial bounded graph without odd cycles. Then, $G_1 \times G_2$ is hyperbolic if and only if G_1 is hyperbolic.*

Proof. Fix some vertex $w_0 \in V(G_1)$ and some edge $[v_1, v_2] \in E(G_2)$.

By Theorem 2.6, there are exactly two components in $G_1 \times G_2$. Since there are no odd cycles, there is no $(w_0, v_1), (w_0, v_2)$ -walk in $G_1 \times G_2$. Thus, let us denote by $(G_1 \times G_2)^1$ the component containing the vertex (w_0, v_1) and by $(G_1 \times G_2)^2$ the component containing the vertex (w_0, v_2) .

Consider $i : V(G_1) \rightarrow V(G_1 \times G_2)^1$ defined as $i(w) := (w, v_1)$ for every $w \in V(G_1)$ such that every w_0, w -walk has even length and $i(w) := (w, v_2)$ for every $w \in V(G_1)$ such that every w_0, w -walk has odd length.

By Proposition 2.3, $d_{G_1 \times G_2}(i(w_1), i(w_2)) = d_{G_1}(w_1, w_2)$ for every $w_1, w_2 \in V(G_1)$ and i is a $(1, 0)$ -quasi-isometric embedding.

Let $(w, v) \in V(G_1 \times G_2)^1$. Let v_j with $j \in \{1, 2\}$ such that every v, v_j -walk has even length. Then, by Proposition 2.3, $d_{G_1 \times G_2}((w, v), (w, v_j)) = d_{G_2}(v, v_j) \leq \text{diam}(G_2)$. Therefore, i is $\text{diam}(G_2)$ -full.

Hence, by Lemma 2.10, there is a $(\text{diam}(G_2) + \frac{1}{2})$ -full $(1, 1)$ -quasi-isometry, $j : G_1 \rightarrow (G_1 \times G_2)^1$, and $(G_1 \times G_2)^1$ is hyperbolic if and only if G_1 is hyperbolic by Theorem 2.9.

The same argument proves that $(G_1 \times G_2)^2$ is hyperbolic. \square

Denote by P_2 the path graph with two vertices, i.e., a graph with two vertices and an edge.

Lemma 2.13. *Let G_1 be a graph with some odd cycle and G_2 a non-trivial bounded graph without odd cycles. Then $G_1 \times G_2$ and $G_1 \times P_2$ are quasi-isometric and $\delta(G_1 \times P_2) \leq \delta(G_1 \times G_2)$.*

Proof. By Theorem 2.6, we know that $G_1 \times G_2$ and $G_1 \times P_2$ are connected graphs.

Denote by v_1 and v_2 the vertices of P_2 and fix $[w_1, w_2] \in E(G_2)$. The map $f : V(G_1 \times P_2) \rightarrow V(G_1 \times [w_1, w_2])$ defined as $f(u, v_j) := (u, w_j)$ for every $u \in V(G_1)$ and $j = 1, 2$, is an isomorphism of graphs; hence, it suffices to prove that $G_1 \times G_2$ and $G_1 \times [w_1, w_2]$ are quasi-isometric.

Consider the inclusion map $i : V(G_1 \times [w_1, w_2]) \rightarrow V(G_1 \times G_2)$. Since $G_1 \times [w_1, w_2]$ is a subgraph of $G_1 \times G_2$, we have $d_{G_1 \times G_2}(x, y) \leq d_{G_1 \times [w_1, w_2]}(x, y)$ for every $x, y \in V(G_1 \times [w_1, w_2])$.

Since G_2 is a graph without odd cycles, every w_1, w_2 -walk has odd length and every w_j, w_j -walk has even length for $j = 1, 2$. Thus Proposition 2.3 gives, for every $x = (u, w_1), y = (v, w_2) \in V(G_1 \times [w_1, w_2])$,

$$d_{G_1 \times [w_1, w_2]}(x, y) = d_{G_1 \times G_2}(x, y) = \min \{L(g) \mid g \text{ is a } u, v\text{-walk of odd length}\}.$$

Furthermore, for every $x = (u, w_j), y = (v, w_j) \in V(G_1 \times [w_1, w_2])$ and $j = 1, 2$,

$$d_{G_1 \times [w_1, w_2]}(x, y) = d_{G_1 \times G_2}(x, y) = \min \{L(g) \mid g \text{ is a } u, v\text{-walk of even length}\}.$$

Hence, $d_{G_1 \times [w_1, w_2]}(x, y) = d_{G_1 \times G_2}(x, y)$ for every $x, y \in V(G_1 \times [w_1, w_2])$, and the inclusion map i is an $(1, 0)$ -quasi-isometric embedding. Therefore, $\delta(G_1 \times P_2) = \delta(G_1 \times [w_1, w_2]) \leq \delta(G_1 \times G_2)$.

Since G_2 is a graph without odd cycles, given any $w \in V(G_2)$, we have either that every w, w_1 -walk has even length and every w, w_2 -walk has odd length or that every w, w_2 -walk has even length and every w, w_1 -walk has odd length. Also, since G_1 is connected, for each $u \in V(G_1)$ there is some $u' \in V(G_1)$ such that $[u, u'] \in E(G_1)$. Therefore, by Proposition 2.3, for every $(u, w) \in V(G_1 \times G_2)$, if $\min \{d_{G_2}(w, w_1), d_{G_2}(w, w_2)\}$ is even, then

$$d_{G_1 \times G_2}((u, w), V(G_1 \times [w_1, w_2])) = d_{G_1 \times G_2}((u, w), V(u \times [w_1, w_2])) = \min \{d_{G_2}(w, w_1), d_{G_2}(w, w_2)\},$$

and if $\min \{d_{G_2}(w, w_1), d_{G_2}(w, w_2)\}$ is odd, then

$$d_{G_1 \times G_2}((u, w), V(G_1 \times [w_1, w_2])) = d_{G_1 \times G_2}((u, w), V(u' \times [w_1, w_2])) = \min \{d_{G_2}(w, w_1), d_{G_2}(w, w_2)\}.$$

In both cases,

$$d_{G_1 \times G_2}((u, w), V(G_1 \times [w_1, w_2])) \leq \text{diam } V(G_2),$$

and i is $(\text{diam } V(G_2))$ -full. By Lemma 2.10, there exists a $(\text{diam } V(G_2) + \frac{1}{2})$ -full $(1, 1)$ -quasi-isometry $g : G_1 \times [w_1, w_2] \rightarrow G_1 \times G_2$. \square

We say that a subgraph Γ of G is *isometric* if $d_\Gamma(x, y) = d_G(x, y)$ for every $x, y \in \Gamma$. It is easy to check that a subgraph Γ of G is isometric if and only if $d_\Gamma(u, v) = d_G(u, v)$ for every $u, v \in V(\Gamma)$. Isometric subgraphs are very important in the study of hyperbolic graphs, as the following result shows.

Lemma 2.14. [45, Lemma 5] *If Γ is an isometric subgraph of G , then $\delta(\Gamma) \leq \delta(G)$.*

A u, v -walk g in G is a *shortcut* of a cycle C if $g \cap C = \{u, v\}$ and $L(g) < d_C(u, v)$ where d_C denotes the length metric on C .

A cycle C' is a *reduction* of the cycle C if both have odd length and C' is the union of a subarc η of C and a shortcut of C joining the endpoints of η . Note that $L(C') \leq L(C) - 2$. We say that a cycle is *minimal* if it has odd length and it does not have a reduction.

Lemma 2.15. *If C is a minimal cycle of G , then $L(C) \leq 4\delta(G)$.*

Proof. We prove first that C is an isometric subgraph of G . Seeking for a contradiction assume that C is not an isometric subgraph. Thus, there exists a shortcut g of C with endpoints u, v . There are two subarcs η_1, η_2 of C joining u and v ; since C has odd length, we can assume that η_1 has even length and η_2 has odd length. If g has even length, then $C' := g \cup \eta_2$ is a reduction of C . If g has odd length, then $C'' := g \cup \eta_1$ is a reduction of C . Hence, C is not minimal, which is a contradiction, and so C is an isometric subgraph of G .

Let $x, y \in C$ with $d_C(x, y) = L(C)/2$ and σ_1, σ_2 the two subarcs of C joining x, y . Since C is an isometric subgraph, $T := \{\sigma_1, \sigma_2\}$ is a geodesic bigon. If p is the midpoint of σ_1 , then Lemma 2.14 gives $L(C)/4 = d_G(p, \{x, y\}) = d_G(p, \sigma_2) \leq \delta(C) \leq \delta(G)$. \square

Given any w_0, w_k -walk $g = [w_0, w_1] \cup [w_1, w_2] \cup \dots \cup [w_{k-1}, w_k]$ in G_1 and $P_2 = [v_1, v_2]$, if $L(g)$ is either odd or even, then we define the $(w_0, v_1), (w_k, v_2)$ -walk for $i \in 1, 2$,

$$\Gamma_1 g := [(w_0, v_1), (w_1, v_2)] \cup [(w_1, v_2), (w_2, v_1)] \cup [(w_2, v_1), (w_3, v_2)] \cup \dots \cup [(w_{k-1}, v_1), (w_k, v_2)],$$

$$\Gamma_1 g := [(w_0, v_1), (w_1, v_2)] \cup [(w_1, v_2), (w_2, v_1)] \cup [(w_2, v_1), (w_3, v_2)] \cup \dots \cup [(w_{k-1}, v_2), (w_k, v_1)],$$

respectively.

Remark 2.16. *By Proposition 2.3, if g is a geodesic path in G_1 , then $\Gamma_1 g$ is a geodesic path in $G_1 \times P_2$.*

Let us define the map $R : V(G_1 \times P_2) \rightarrow V(G_1 \times P_2)$ as $R(w, v_1) = (w, v_2)$ and $R(w, v_2) = (w, v_1)$ for every $w \in V(G_1)$, and the path $\Gamma_2 g$ as $\Gamma_2 g = R(\Gamma_1 g)$.

Let us define the map $(\Gamma_1 g)' : g \rightarrow \Gamma_1 g$ which is an isometry on the edges and such that $(\Gamma_1 g)'(w_j) = (w_j, v_1)$ if j is even and $(\Gamma_1 g)'(w_j) = (w_j, v_2)$ if j is odd. Also, let $(\Gamma_2 g)' : g \rightarrow \Gamma_2 g$ be the map defined by $(\Gamma_2 g)' := R \circ (\Gamma_1 g)'$.

Given a graph G , denote by $\mathfrak{C}(G)$ the set of minimal cycles of G .

Lemma 2.17. *Let G_1 be a graph with some odd cycle and $P_2 = [v_1, v_2]$. Consider a geodesic $g = [w_0 w_k] = [w_0, w_1] \cup [w_1, w_2] \cup \dots \cup [w_{k-1}, w_k]$ in G_1 . Let us define $w'_0 := (\Gamma_1 g)'(w_0) = (w_0, v_1)$ and $w'_k := (\Gamma_2 g)'(w_k)$, i.e., $w'_k := (w_k, v_1)$ or $w'_k := (w_k, v_2)$ if k is odd or even, respectively. Then $d_{G_1 \times P_2}(w'_0, w'_k) > \sqrt{d_{G_1}(w_j, \mathfrak{C}(G_1))}$ for every $0 \leq j \leq k$.*

Proof. Fix $0 \leq j \leq k$. Define

$$\mathfrak{P} := \{\sigma \mid \sigma \text{ is a } w_0, w_k\text{-walk such that } L(\sigma) \text{ has a parity different from that of } k\}.$$

Proposition 2.3 gives

$$d_{G_1 \times P_2}(w'_0, w'_k) = \min \{L(\sigma) \mid \sigma \in \mathfrak{P}\}.$$

Choose $\sigma_0 \in \mathfrak{P}$ such that $L(\sigma_0) = d_{G_1 \times P_2}(w'_0, w'_k)$. Since $L(g) + L(\sigma_0)$ is odd, we have $L(g) + L(\sigma_0) = 2t + 1$ for some positive integer t . Thus $d_{G_1 \times P_2}(w'_0, w'_k) = L(\sigma_0) > \frac{1}{2}(2t + 1)$.

If $g \cup \sigma_0$ is a cycle, then let us define $C_0 := g \cup \sigma_0$. Thus, $L(C_0) = 2t + 1$ and $d_{G_1}(w_j, C_0) = 0$ for every $0 \leq j \leq k$. Otherwise, we may assume that $g \cap \sigma_0 = [w_0 w_{i_1}] \cup [w_{i_2} w_k]$ for some $0 \leq i_1 < i_2 \leq k$. If

$\sigma_1 = \sigma_0 \setminus g$, then let us define $C_0 := [w_{i_1} w_{i_2}] \cup \sigma_1$ (where $[w_{i_1} w_{i_2}] \subset g$). Hence, C_0 is a cycle, $L(C_0) \leq 2t - 1$ and $d_{G_1}(w_j, C_0) < \frac{1}{2}(2t + 1)$.

If C_0 is not minimal, then consider a reduction C_1 of C_0 . Let us repeat the process until we obtain a minimal cycle C_s . Note that $L(C_1) \leq L(C_0) - 2$ and for every point $p_1 \in C_0$, $d_{G_1}(p_1, C_1) < \frac{1}{2}L(C_0)$. Now, repeating the argument, for every $1 < i \leq s$, $L(C_i) \leq L(C_{i-1}) - 2$ and for every point $p_i \in C_{i-1}$, $d_{G_1}(p_i, C_i) < \frac{1}{2}L(C_{i-1})$. Therefore,

$$\begin{aligned} d_{G_1}(w_j, \mathfrak{C}(G_1)) &\leq d_{G_1}(w_j, C_s) \leq d_{G_1}(w_j, C_0) + \frac{1}{2}L(C_0) + \frac{1}{2}L(C_1) + \cdots + \frac{1}{2}L(C_s) \\ &< \frac{1}{2}(2t + 1) + \frac{1}{2}(2t - 1) + \cdots + \frac{5}{2} + \frac{3}{2}. \end{aligned}$$

Hence,

$$d_{G_1}(w_j, \mathfrak{C}(G_1)) < \frac{1}{2} \sum_{i=1}^t (2i + 1) = \frac{1}{2}t^2 + t < \left(\frac{1}{2}(2t + 1)\right)^2 < \left(d_{G_1 \times P_2}(w'_0, w'_k)\right)^2.$$

□

Corollary 2.18. *Let G_1 be a hyperbolic graph with some odd cycle and $P_2 = [v_1, v_2]$. Consider a geodesic $g = [w_0 w_k] = [w_0, w_1] \cup [w_1, w_2] \cup \cdots \cup [w_{k-1}, w_k]$ in G_1 . Let us define $w'_0 := (\Gamma_1 g)'(w_0) = (w_0, v_1)$ and $w'_k := (\Gamma_2 g)'(w_k)$. Then, we have for every $0 \leq j \leq k$,*

$$\frac{1}{2} \left(k + \sqrt{d_{G_1}(w_j, \mathfrak{C}(G_1))} \right) \leq d_{G_1 \times P_2}(w'_0, w'_k) \leq k + 2d_{G_1}(w_j, \mathfrak{C}(G_1)) + 4\delta(G_1).$$

Proof. Corollary 2.5 and Lemma 2.17 give $d_{G_1 \times P_2}(w'_0, w'_k) \geq k$ and $d_{G_1 \times P_2}(w'_0, w'_k) \geq \sqrt{d_{G_1}(w_j, \mathfrak{C}(G_1))}$, and these inequalities provide the lower bound of $d_{G_1 \times P_2}(w'_0, w'_k)$.

Consider a geodesic γ joining w_j and $C \in \mathfrak{C}(G_1)$ with $L(\gamma) = d_{G_1}(w_j, C) = d_{G_1}(w_j, \mathfrak{C}(G_1))$ and the w_0, w_k -walk

$$g' := [w_0 w_j] \cup \gamma \cup C \cup \gamma \cup [w_j w_k].$$

One can check that $\Gamma_1 g'$ is a w'_0, w'_k -walk in $G_1 \times P_2$, and so Lemma 2.15 gives

$$d_{G_1 \times P_2}(w'_0, w'_k) \leq L(\Gamma_1 g') = L(g') = k + 2d_{G_1}(w_j, \mathfrak{C}(G_1)) + L(C) \leq k + 2d_{G_1}(w_j, \mathfrak{C}(G_1)) + 4\delta(G_1).$$

□

If $[v_1, v_2] \in E(G)$, then we say that the point $x \in [v_1, v_2]$ with $d_G(x, v_1) = d_G(x, v_2) = 1/2$ is the *midpoint* of $[v_1, v_2]$. Denote by $J(G)$ the set of vertices and midpoints of edges in G . Consider the set $\mathbb{T}_1(G)$ of geodesic triangles T in G that are cycles and such that the three vertices of the triangle T belong to $J(G)$, and denote by $\delta_1(G)$ the infimum of the constants λ such that every triangle in $\mathbb{T}_1(G)$ is λ -thin.

The following three results, which appear in [8], will be used throughout the paper.

Theorem 2.19. [8, Theorem 2.5] *For every graph G we have $\delta_1(G) = \delta(G)$.*

The next result will narrow the posible values for the hyperbolicity constant δ .

Theorem 2.20. [8, Theorem 2.6] *Let G be any graph. Then $\delta(G)$ is always a multiple of $1/4$.*

Theorem 2.21. [8, Theorem 2.7] *For any hyperbolic graph G , there exists a geodesic triangle $T \in \mathbb{T}_1(G)$ such that $\delta(T) = \delta(G)$.*

Consider the set $\mathbb{T}_v(G)$ of geodesic triangles T in G that are cycles and such that the three vertices of the triangle T belong to $V(G)$, and denote by $\delta_v(G)$ the infimum of the constants λ such that every triangle in $\mathbb{T}_v(G)$ is λ -thin.

Theorem 2.22. *For every graph G we have $\delta_v(G) \leq \delta(G) \leq 4\delta_v(G) + 1/2$. Hence, G is hyperbolic if and only if $\delta_v(G) < \infty$. Furthermore, if G is hyperbolic, then $\delta_v(G)$ is always a multiple of $1/2$ and there exist a geodesic triangle $T = \{x, y, z\} \in \mathbb{T}_v(G)$ and $p \in [xy] \cap J(G)$ such that $d(p, [xz] \cup [zy]) = \delta(T) = \delta_v(G)$.*

Proof. The inequality $\delta_v(G) \leq \delta(G)$ is direct.

Consider the set $\mathbb{T}'_v(G)$ of geodesic triangles T in G such that the three vertices of the triangle T belong to $V(G)$, and denote by $\delta'_v(G)$ the infimum of the constants λ such that every triangle in $\mathbb{T}'_v(G)$ is λ -thin. The argument in the proof of [46, Lemma 2.1] gives that $\delta'_v(G) = \delta_v(G)$.

In order to prove the upper bound of $\delta(G)$, assume first that G is hyperbolic. We can assume $\delta'_v(G) < \infty$, since otherwise the inequality is direct. By Theorem 2.21, there exists a geodesic triangle $T = \{x, y, z\}$ that is a cycle with $x, y, z \in J(G)$ and $p \in [xy]$ such that $d(p, [xz] \cup [zy]) = \delta(T) = \delta(G)$. Assume that $x, y, z \in J(G) \setminus V(G)$ (otherwise, the argument is simpler). Let $x_1, x_2, y_1, y_2, z_1, z_2 \in T \cap V(G)$ such that $x \in [x_1, x_2]$, $y \in [y_1, y_2]$, $z \in [z_1, z_2]$ and $x_2, y_1 \in [xy]$, $y_2, z_1 \in [yz]$, $z_2, x_1 \in [xz]$. Since $H := \{x_2, y_1, y_2, z_1, z_2, x_1\}$ is a geodesic hexagon with vertices in $V(G)$, it is $4\delta'_v(G)$ -thin and every point $w \in [y_1, y_2] \cup [y_2 z_1] \cup [z_1, z_2] \cup [z_2 x_1] \cup [x_1, x_2]$ verifies $d(w, [xz] \cup [zy]) \leq 1/2$, we have

$$\delta(G) = d(p, [xz] \cup [zy]) \leq d(p, [y_1, y_2] \cup [y_2 z_1] \cup [z_1, z_2] \cup [z_2 x_1] \cup [x_1, x_2]) + 1/2 \leq 4\delta'_v(G) + 1/2 = 4\delta_v(G) + 1/2.$$

Assume now that G is not hyperbolic. Therefore, for each $M > 0$ there exists a geodesic triangle $T = \{x, y, z\}$ that is a cycle with $x, y, z \in J(G)$ and $p \in [xy]$ such that $d(p, [xz] \cup [zy]) \geq M$. The previous argument gives $M \leq 4\delta_v(G) + 1/2$ and, since M is arbitrary, we deduce $\delta_v(G) = \infty = \delta(G)$.

Finally, consider any geodesic triangle $T = \{x, y, z\}$ in $\mathbb{T}_v(G)$. Since $d(p, [xz] \cup [zy]) = d(p, ([xz] \cup [zy]) \cap V(G))$, $d(p, [xz] \cup [zy])$ attains its maximum value when $p \in J(G)$. Hence, $\delta(T)$ is a multiple of $1/2$ for every geodesic triangle $T \in \mathbb{T}_v(G)$. Since the set of non-negative numbers that are multiple of $1/2$ is a discrete set, if G is hyperbolic, then $\delta(G)$ is a multiple of $1/2$ and there exist a geodesic triangle $T = \{x, y, z\} \in \mathbb{T}_v(G)$ and $p \in [xy] \cap J(G)$ such that $d(p, [xz] \cup [zy]) = \delta(T) = \delta_v(G)$. This finishes the proof. \square

Theorem 2.23. *If G_1 is a non-hyperbolic graph, then $G_1 \times P_2$ is not hyperbolic.*

Proof. Since G_1 is not hyperbolic, by Theorem 2.22, given any $R > 0$ there is a geodesic triangle $T = \{x, y, z\}$ that is a cycle, with $x, y, z \in V(G_1)$ and such that T is not R -thin. Therefore, there exists some point $m \in T$, let us assume that $m \in [xy]$, such that $d_{G_1}(m, [yz] \cup [zx]) > R$.

Seeking for a contradiction let us assume that $G_1 \times P_2$ is δ -hyperbolic.

Suppose that for some $R > \delta$, there is a geodesic triangle $T = \{x, y, z\}$ that is an even cycle in G_1 , with $x, y, z \in V(G_1)$ and such that T is not R -thin. Consider the (closed) path $\Lambda = [xy] \cup [yz] \cup [zx]$. Then, since T has even length, the path $\Gamma_1 \Lambda$ defines a cycle in $G_1 \times P_2$. Let $\gamma_1, \gamma_2, \gamma_3$ be the paths in $\Gamma_1 \Lambda$ corresponding to $[xy], [yz], [zx]$, respectively. By Corollary 2.5, the curves γ_1, γ_2 and γ_3 are geodesics, and $d_{G_1 \times P_2}((\Gamma_1 \Lambda)'(m), \gamma_2 \cup \gamma_3) > \delta$, leading to contradiction.

Suppose that for every $R > 0$, there is a geodesic triangle $T = \{x, y, z\}$ which is an odd cycle, with $x, y, z \in V(G_1)$ and such that T is not R -thin.

Let $T_1 = \{x, y, z\}$ be a geodesic triangle as above and let us assume that $\text{diam}(T_1) = D > 8\delta$.

Let $T_2 = \{x', y', z'\}$ be another geodesic triangle as above such that T_2 is not $3(D + 8\delta)$ -thin, this is, there is a point m in one of the sides, let us call it σ , of T_2 such that $d_{G_1}(m, T_2 \setminus \sigma) > 3(D + 8\delta)$.

Let $g = [w_0 w_k]$ with $w_0 \in T_1$ and $w_k \in T_2$ be a shortest geodesic in G_1 joining T_1 and T_2 (if T_1 and T_2 intersect, just assume that g is a single vertex, $w_0 = w_k$, in the intersection).

Let us assume that $w_0 \in [xz]$ and $w_k \in [x'z']$. Then, let us consider the cycle C in G_1 given by the union of the geodesics in T_1 , g , the geodesics in T_2 and the inverse of g from w_k to w_0 , this is,

$$C := [w_0 x] \cup [xy] \cup [yz] \cup [zw_0] \cup [w_0 w_k] \cup [w_k x'] \cup [x' y'] \cup [y' z'] \cup [z' w_k] \cup [w_k w_0].$$

Since T_1, T_2 are odd cycles, C is an even cycle. Therefore, $\Gamma_1 C$ defines a cycle in $G_1 \times P_2$. Moreover, by Remark 2.16, $\Gamma_1 C$ is a geodesic decagon in $G_1 \times P_2$ with sides $\gamma_1 = (\Gamma_1 C)'([w_0 x])$, $\gamma_2 = (\Gamma_1 C)'([xy])$, $\gamma_3 = (\Gamma_1 C)'([yz])$, $\gamma_4 = (\Gamma_1 C)'([zw_0])$, $\gamma_5 = (\Gamma_1 C)'([w_0 w_k])$, $\gamma_6 = (\Gamma_1 C)'([w_k x'])$, $\gamma_7 = (\Gamma_1 C)'([x' y'])$, $\gamma_8 = (\Gamma_1 C)'([y' z'])$, $\gamma_9 = (\Gamma_1 C)'([z' w_k])$ and $\gamma_{10} = (\Gamma_1 C)'([w_k w_0])$.

Since we are assuming that $G_1 \times P_2$ is δ -hyperbolic, then for every $1 \leq i \leq 10$ and every point $p \in \gamma_i$, $d_{G_1 \times P_2}(p, C \setminus \gamma_i) \leq 8\delta$.

Let $p := (\Gamma_1 C)'(m)$.

Case 1. Suppose that $d_{G_1}(m, T_1 \cup g) > 8\delta$.

By assumption, $d_{G_1}(m, T_2 \setminus \sigma) > 8\delta$. If $\sigma = [x'y']$ (resp. $\sigma = [y'z']$), then $p \in \gamma_7$ (resp. $p \in \gamma_8$) and, by Corollary 2.5, $d_{G_1 \times P_2}(p, C \setminus \gamma_7) > 8\delta$ (resp. $d_{G_1 \times P_2}(p, C \setminus \gamma_8) > 8\delta$) leading to contradiction. If $\sigma = [x'z']$, since $[x'z'] = [x'w_k] \cup [w_kz']$, let us assume $m \in [x'w_k]$. Then, since $d_{G_1}(m, w_k) > 8\delta$, it follows that $d_{G_1}(m, [w_kz']) > 8\delta$. Thus, $p \in \gamma_6$ and, by Corollary 2.5, $d_{G_1 \times P_2}(p, C \setminus \gamma_6) > 8\delta$ leading to contradiction.

Case 2. Suppose that $d_{G_1}(m, T_1 \cup g) \leq 8\delta$ and $L(g) \leq 8\delta$. Then, for every point q in $T_1 \cup g$, $d_{G_1}(m, q) \leq 8\delta + D + 8\delta$. In particular, $d_{G_1}(m, w_k) \leq 8\delta + D + 8\delta$. Therefore, $m \in [x'z']$ and let us assume that $m \in [x'w_k]$. Since $d_{G_1}(m, x') \geq d_{G_1}(m, [x'y'] \cup [y'z']) > 3(D + 8\delta)$, there is a point $m' \in [x'm] \subset [x'w_k]$ such that $d_{G_1}(m, m') = 2(D + 8\delta)$. Then, $d_{G_1}(m', T_1 \cup g) \geq 2(D + 8\delta) - D - 8\delta - 8\delta = D > 8\delta$. Also, it is trivial to check that $d_{G_1}(m', [x'y'] \cup [y'z']) > 3(D + 8\delta) - 2(D + 8\delta) > 8\delta$ and since $[x'z']$ is a geodesic, $d_{G_1}(m', [z'w_k]) > 8\delta$. Thus, if $p' := (\Gamma_1 C)'(m')$, then $p' \in \gamma_6$ and, by Corollary 2.5, $d_{G_1 \times P_2}(p', C \setminus \gamma_6) > 8\delta$ leading to contradiction.

Case 3. Suppose that $d_{G_1}(m, T_1 \cup g) \leq 8\delta$ and $L(g) > 8\delta$. Since g is a shortest geodesic in G_1 joining T_1 and T_2 , this implies that $d_{G_1}(T_1, T_2) > 8\delta$ and $d_{G_1}(m, [w_0w_k]) \leq 8\delta$. Moreover, $d_{G_1}(m, w_k) \leq 16\delta$. Otherwise, there is a point $q \in [w_0w_k]$ such that $d_{G_1}(m, q) \leq 8\delta$ and $d_{G_1}(q, w_k) > 8\delta$ which means that $d_{G_1}(q, w_0) < d_{G_1}(w_0, w_k) - 8\delta$ and $d_{G_1}(m, w_0) < d_{G_1}(w_0, w_k)$ leading to contradiction.

Since $d_{G_1}(m, w_k) \leq 16\delta$, $m \in [x'z']$. Let us assume that $m \in [x'w_k]$. Since $d_{G_1}(m, [x'y'] \cup [y'z']) > 3(D + 8\delta)$, there is a point $m' \in [x'm] \subset [x'w_k]$ such that $d_{G_1}(m, m') = 2(D + 8\delta)$. Let us see that $d_{G_1}(m', [w_0w_k]) > 8\delta$. Suppose there is some $q \in [w_0w_k]$ such that $d_{G_1}(m', q) \leq 8\delta$. Since $m' \in T_2$ and g is a shortest geodesic joining T_1 and T_2 , $d_{G_1}(q, w_k) \leq 8\delta$. However, $32\delta < 2(D + 8\delta) = d_{G_1}(m', m) \leq d_{G_1}(m', q) + d_{G_1}(q, w_k) + d_{G_1}(w_k, m) \leq 8\delta + 8\delta + 16\delta$ which is a contradiction. Hence, $d_{G_1}(m', [w_0w_k]) > 8\delta$. Also, it is trivial to check that $d_{G_1}(m', [x'y'] \cup [y'z']) > 3(D + 8\delta) - 2(D + 8\delta) > 8\delta$ and since $[x'z']$ is a geodesic, $d_{G_1}(m', [z'w_k]) > 8\delta$. Thus, if $p' := (\Gamma_1 C)'(m')$, then $p' \in \gamma_6$ and, by Corollary 2.5, $d_{G_1 \times P_2}(p', C \setminus \gamma_6) > 8\delta$ leading to contradiction. \square

Proposition 2.8, Lemma 2.13 and Theorems 2.11, 2.12 and 2.23 have the following consequence.

Corollary 2.24. *If G_1 is a non-hyperbolic graph and G_2 is some non-trivial graph, then $G_1 \times G_2$ is not hyperbolic.*

Proposition 2.8 and Corollary 2.24 provide a necessary condition for the hyperbolicity of $G_1 \times G_2$.

Theorem 2.25. *Let G_1, G_2 be non-trivial graphs. If $G_1 \times G_2$ is hyperbolic, then one factor graph is hyperbolic and the other one is bounded.*

Theorems 2.11 and 2.12 show that this necessary condition is also sufficient if either G_2 has some odd cycle or G_1 and G_2 do not have odd cycles (when G_1 is a hyperbolic graph and G_2 is a bounded graph). We deal now with the other case, when G_1 has some odd cycle and G_2 does not have odd cycles.

Theorem 2.26. *Let G_1 be a graph with some odd cycle and G_2 a non-trivial bounded graph without odd cycles. Assume that G_1 satisfies the following property: for each $M > 0$ there exist a geodesic g joining two minimal cycles of G_1 and a vertex $u \in g \cap V(G_1)$ with $d_{G_1}(u, \mathfrak{C}(G_1)) \geq M$. Then $G_1 \times G_2$ is not hyperbolic.*

Proof. If G_1 is not hyperbolic, then Corollary 2.24 gives that $G_1 \times G_2$ is not hyperbolic. Assume now that G_1 is hyperbolic. By Theorem 2.9 and Lemma 2.13, we can assume that $G_2 = P_2$ and $V(P_2) = \{v_1, v_2\}$.

Fix $M > 0$ and choose a geodesic $g = [w_0w_k] = [w_0, w_1] \cup [w_1, w_2] \cup \dots \cup [w_{k-1}, w_k]$ joining two minimal cycles in G_1 and $0 < r < k$ with $d_{G_1}(w_r, \mathfrak{C}(G_1)) \geq M$.

Define the paths g_1 and g_2 in $G_1 \times P_2$ as $g_1 := \Gamma_1 g$ and $g_2 := \Gamma_2 g$. Since $L(g_1) = L(g_2) = L(g) = d_{G_1}(w_0, w_k)$, we have

$$d_{G_1 \times P_2}(g_1(w_0), g_1(w_k)) \leq L(g_1) = d_{G_1}(w_0, w_k), \quad d_{G_1 \times P_2}(g_2(w_0), g_2(w_k)) \leq L(g_2) = d_{G_1}(w_0, w_k).$$

Corollary 2.5 gives that

$$d_{G_1 \times P_2}(g_1(w_0), g_1(w_k)) \geq d_{G_1}(w_0, w_k), \quad d_{G_1 \times P_2}(g_2(w_0), g_2(w_k)) \geq d_{G_1}(w_0, w_k).$$

Hence, g_1 and g_2 are geodesics in $G_1 \times P_2$. Choose geodesics $g_3 = [g_1(w_0)g_2(w_0)]$ and $g_4 = [g_1(w_k)g_2(w_k)]$ in $G_1 \times P_2$. Since $d_{P_2}(v_1, v_2) = 1$ is odd, Proposition 2.3 gives

$$\begin{aligned} d_{G_1 \times P_2}(g_1(w_0), g_2(w_0)) &= \min \{L(\sigma) \mid \sigma \text{ is a } w_0, w_0\text{-walk}\} \\ &= \min \{L(\sigma) \mid \sigma \text{ cycle of odd length containing } w_0\}. \end{aligned}$$

Since w_0 belongs to a minimal cycle, $L(g_3) \leq 4\delta(G_1)$ by Lemma 2.15. In a similar way, we obtain $L(g_4) \leq 4\delta(G_1)$.

Consider the geodesic quadrilateral $Q := \{g_1, g_2, g_3, g_4\}$ in $G_1 \times P_2$. Thus $d_{G_1 \times P_2}(g_1(w_r), g_2 \cup g_3 \cup g_4) \leq 2\delta(G_1 \times P_2)$. Since $\max \{L(g_3), L(g_4)\} \leq 4\delta(G_1)$, we deduce $d_{G_1 \times P_2}(g_1(w_r), g_2) \leq 2\delta(G_1 \times P_2) + 4\delta(G_1)$.

Let $0 \leq j \leq k$ with $d_{G_1 \times P_2}(g_1(w_r), g_2) = d_{G_1 \times P_2}(g_1(w_r), g_2(w_j))$. Let us define $w'_r := g_1(w_r)$ and $w'_j := g_2(w_j)$. Thus Lemma 2.17 gives

$$\sqrt{M} \leq \sqrt{d_{G_1}(w_r, \mathfrak{C}(G_1))} \leq d_{G_1 \times P_2}(w'_r, w'_j) = d_{G_1 \times P_2}(w'_r, g_2) \leq 2\delta(G_1 \times P_2) + 4\delta(G_1),$$

and since M is arbitrarily large, we deduce that $G_1 \times P_2$ is not hyperbolic. \square

Lemma 2.27. *Let G_1 be a hyperbolic graph and suppose there is some constant $K > 0$ such that for every vertex $w \in G_1$, $d_{G_1}(w, \mathfrak{C}(G_1)) \leq K$. Then, $G_1 \times P_2$ is hyperbolic.*

Proof. Denote by v_1 and v_2 the vertices of P_2 . Let $i : V(G_1) \rightarrow V(G_1 \times P_2)$ defined as $i(w) := (w, v_1)$ for every $w \in G_1$.

For every pair of vertices $x, y \in V(G_1)$, by Corollary 2.5, $d_{G_1}(x, y) \leq d_{G_1 \times P_2}(i(x), i(y))$. By Corollary 2.18,

$$d_{G_1 \times P_2}(i(x), i(y)) \leq d_{G_1}(x, y) + 2d_{G_1}(x, \mathfrak{C}(G_1)) + 4\delta(G_1) \leq d_{G_1}(x, y) + 2K + 4\delta(G_1).$$

Therefore, $i : V(G_1) \rightarrow V(G_1 \times P_2)$ is a $(1, 2K + 4\delta(G_1))$ -quasi-isometric embedding.

Notice that for every $(w, v_1) \in V(G_1 \times P_2)$, $(w, v_1) = i(w)$. Also, for any $(w, v_2) \in V(G_1 \times P_2)$, since G_1 is connected, there is some edge $[w, w'] \in E(G_1)$ and we have $[(w, v_2), (w', v_1)] \in E(G_1 \times P_2)$. Therefore, $i : V(G_1) \rightarrow V(G_1 \times P_2)$ is 1-full.

Thus, by Lemma 2.10, G_1 and $G_1 \times P_2$ are quasi-isometric and, by Theorem 2.9, $G_1 \times P_2$ is hyperbolic. \square

Theorem 2.11 and Lemmas 2.13 and 2.27 have the following consequence.

Theorem 2.28. *Let G_1 be a hyperbolic graph and G_2 some non-trivial bounded graph. If there is some constant $K > 0$ such that for every vertex $w \in G_1$, $d_{G_1}(w, \mathfrak{C}(G_1)) \leq K$, then $G_1 \times G_2$ is hyperbolic.*

We will finish this section with a characterization of the hyperbolicity of $G_1 \times G_2$, under an additional hypothesis. Since the proof of this result is long and technical, in order to make the arguments more transparent, we collect some results we need along the proof in technical lemmas.

Let J be a finite or infinite index set. Now, given a graph G_1 , we define some graphs related to G_1 which will be useful in the following results. Let $B_j := B_{G_1}(w_j, K_j)$ with $w_j \in V(G_1)$ and $K_j \in \mathbb{Z}^+$, for any $j \in J$, such that $\sup_j K_j = K < \infty$, $\overline{B}_{j_1} \cap \overline{B}_{j_2} = \emptyset$ if $j_1 \neq j_2$, and every odd cycle C in G_1 satisfies $C \cap B_j \neq \emptyset$ for some $j \in J$. Denote by G'_1 the subgraph of G_1 induced by $V(G_1) \setminus (\cup_j B_j)$. Let $N_j := \partial B_j = \{w \in V(G_1) : d_{G_1}(w, w_j) = K_j\}$. Denote by G_1^* the graph with $V(G_1^*) = V(G'_1) \cup (\cup_j \{w_j^*\})$, where w_j^* are additional vertices, and $E(G_1^*) = E(G'_1) \cup (\cup_j \{[w, w_j^*] : w \in N_j\})$. We have $G'_1 = G_1 \cap G_1^*$.

Lemma 2.29. *Let G_1 be a graph as above. Then, there exists a quasi-isometry $g : G_1 \rightarrow G_1^*$ with $g(w_j) = w_j^*$ for every $j \in J$.*

Proof. Let $f : V(G_1) \rightarrow V(G_1^*)$ defined as $f(u) = u$ for every $u \in V(G'_1)$, and $f(u) = w_i^*$ for every $u \in V(B_i)$. It is clear that $f : V(G_1) \rightarrow V(G_1^*)$ is 0-full.

Now, we focus on proving that $f : V(G_1) \rightarrow V(G_1^*)$ is a $(K, 2K)$ -quasi-isometric embedding. For every $u, v \in V(G_1)$, it is clear that $d_{G_1^*}(f(u), f(v)) \leq d_{G_1}(u, v)$.

In order to prove the other inequality, let us fix $u, v \in V(G_1)$ and let us consider a geodesic γ in G_1^* joining $f(u)$ and $f(v)$.

Assume that $u, v \in V(G_1')$. If $L(\gamma) = d_{G_1}(u, v)$, then $d_{G_1}(u, v) = d_{G_1^*}(f(u), f(v))$. If $L(\gamma) < d_{G_1}(u, v)$, then γ meets some w_j^* . Since γ is a compact set, it intersects just a finite number of w_j^* 's, which we denote by $w_{j_1}^*, \dots, w_{j_r}^*$. We consider γ as an oriented curve from $f(u)$ to $f(v)$; thus we can assume that γ meets $w_{j_1}^*, \dots, w_{j_r}^*$ in this order.

Let us define the following vertices in γ

$$w_i^1 = [f(u)w_{j_i}^*] \cap N_{j_i}, \quad w_i^2 = [w_{j_i}^*f(v)] \cap N_{j_i},$$

for every $1 \leq i \leq r$. Note that $[w_i^2 w_{i+1}^1] \subset G_1'$ for every $1 \leq i < r$ (it is possible to have $w_i^2 = w_{i+1}^1$).

Since $d_{G_1^*}(w_i^1, w_i^2) = 2$ and $d_{G_1}(w_i^1, w_i^2) \leq 2K$, we have $d_{G_1^*}(w_i^1, w_i^2) \geq \frac{1}{K} d_{G_1}(w_i^1, w_i^2)$ for every $1 \leq i \leq r$. Thus,

$$\begin{aligned} d_{G_1^*}(f(u), f(v)) &= d_{G_1^*}(f(u), w_1^1) + \sum_{i=1}^r d_{G_1^*}(w_i^1, w_i^2) + \sum_{i=1}^{r-1} d_{G_1^*}(w_i^2, w_{i+1}^1) + d_{G_1^*}(w_r^2, f(v)) \\ &\geq d_{G_1}(u, w_1^1) + \frac{1}{K} \sum_{i=1}^r d_{G_1}(w_i^1, w_i^2) + \sum_{i=1}^{r-1} d_{G_1}(w_i^2, w_{i+1}^1) + d_{G_1}(w_r^2, v) \\ &\geq \frac{1}{K} \left(d_{G_1}(u, w_1^1) + \sum_{i=1}^r d_{G_1}(w_i^1, w_i^2) + \sum_{i=1}^{r-1} d_{G_1}(w_i^2, w_{i+1}^1) + d_{G_1}(w_r^2, v) \right) \\ &\geq \frac{1}{K} d_{G_1}(u, v). \end{aligned}$$

Assume that $f(u) = f(v)$. Therefore, there exists j with $u, v \in B_j$ and

$$d_{G_1^*}(f(u), f(v)) = 0 > d_{G_1}(u, v) - 2K.$$

Assume now that u and/or v does not belong to $V(G_1')$ and $f(u) \neq f(v)$. Let u_0, v_0 be the closest vertices in $V(G_1') \cap \gamma$ to $f(u), f(v)$, respectively (it is possible to have $u_0 = f(u)$ or $v_0 = f(v)$). Since $u_0, v_0 \in V(G_1')$, $u_0 = f(u_0), v_0 = f(v_0)$, we have $d_{G_1}(u, u_0) < 2K$ and $d_{G_1}(v, v_0) < 2K$. Hence,

$$\begin{aligned} d_{G_1^*}(f(u), f(v)) &= d_{G_1^*}(f(u), u_0) + d_{G_1^*}(u_0, v_0) + d_{G_1^*}(v_0, f(v)) \\ &\geq d_{G_1^*}(f(u_0), f(v_0)) \\ &\geq \frac{1}{K} d_{G_1}(u_0, v_0) \\ &\geq \frac{1}{K} \left(d_{G_1}(u, v) - d_{G_1}(u, u_0) - d_{G_1}(v, v_0) \right) \\ &> \frac{1}{K} d_{G_1}(u, v) - 4. \end{aligned}$$

If $K \geq 2$, then $d_{G_1^*}(f(u), f(v)) > \frac{1}{K} d_{G_1}(u, v) - 2K$. If $K = 1$, then $d_{G_1}(u, u_0) \leq 1, d_{G_1}(v, v_0) \leq 1$, and $d_{G_1^*}(f(u), f(v)) \geq d_{G_1}(u, v) - 2$.

Finally, we conclude that $f : V(G_1) \rightarrow V(G_1^*)$ is a $(K, 2K)$ -quasi-isometric embedding. Thus, Lemma 2.10 provides a quasi-isometry $g : G_1 \rightarrow G_1^*$ with the required property. \square

Definition 2.30. Given a graph G_1 and some index set J let $\mathcal{B}_J = \{B_j\}_{j \in J}$ be a family of balls where $B_j := B_{G_1}(w_j, K_j)$ with $w_j \in V(G_1)$, $K_j \in \mathbb{Z}^+$ for any $j \in J$, $\sup_j K_j = K < \infty$ and $\overline{B}_{j_1} \cap \overline{B}_{j_2} = \emptyset$ if $j_1 \neq j_2$. Suppose that every odd cycle C in G_1 satisfies that $C \cap B_j \neq \emptyset$ for some $j \in J$. If there is some constant $M > 0$ such that for every $j \in J$, there is an odd cycle C_j such that $C_j \cap B_j \neq \emptyset$ with $L(C_j) < M$, then we say that \mathcal{B}_J is M -regular.

Remark 2.31. If J is finite, then there exists $M > 0$ such that $\{B_j\}_{j \in J}$ is M -regular.

Denote by G^* the graph with $V(G^*) = V(G'_1 \times P_2) \cup (\cup_j \{w_j^*\})$, where G'_1 is a graph as above and w_j^* are additional vertices, and $E(G^*) = E(G'_1 \times P_2) \cup (\cup_j \{[w, w_j^*] : \pi_1(w) \in N_j\})$.

Lemma 2.32. *Let G_1 be a graph as above and P_2 with $V(P_2) = \{v_1, v_2\}$. If G_1 is hyperbolic and \mathcal{B}_J as above is M -regular, then there exists a quasi-isometry $f : G_1 \times P_2 \rightarrow G^*$ with $f(w_j, v_i) = w_j^*$ for every $j \in J$ and $i \in \{1, 2\}$.*

Proof. Let $F : V(G_1 \times P_2) \rightarrow V(G^*)$ defined as $F(v, v_i) = (v, v_i)$ for every $v \in V(G'_1)$, and $F(v, v_i) = w_j^*$ for every $v \in V(B_j)$. It is clear that $F : V(G_1 \times P_2) \rightarrow V(G^*)$ is 0-full. Recall that we denote by $\pi_1 : G_1 \times P_2 \rightarrow G_1$ the projection map. Define $\pi^* : G^* \rightarrow G_1$ as $\pi^* = \pi_1$ on $G'_1 \times P_2$ and $\pi^*(x) = w_j$ for every x with $d_{G^*}(x, w_j^*) < 1$ for some $j \in J$.

Now, we focus on proving that $F : V(G_1 \times P_2) \rightarrow V(G^*)$ is a quasi-isometric embedding. For every $(w, v_i), (w', v_{i'}) \in V(G_1 \times P_2)$, one can check

$$d_{G^*}(F(w, v_i), F(w', v_{i'})) \leq d_{G_1 \times P_2}((w, v_i), (w', v_{i'})).$$

In order to prove the other inequality, let us fix $(w, v_i), (w', v_{i'}) \in V(G'_1 \times P_2)$ (the inequalities in other cases can be obtained from the one in this case, as in the proof of Lemma 2.29). Consider a geodesic $\gamma := [F(w, v_i)F(w', v_{i'})]$ in G^* . If $L(\gamma) = d_{G_1 \times P_2}((w, v_i), (w', v_{i'}))$, then

$$d_{G^*}(F(w, v_i), F(w', v_{i'})) = d_{G_1 \times P_2}((w, v_i), (w', v_{i'})).$$

If $L(\gamma) < d_{G_1 \times P_2}((w, v_i), (w', v_{i'}))$, then $\pi^*(\gamma)$ meets some B_j . Since γ is a compact set, $\pi^*(\gamma)$ intersects just a finite number of B_j 's, which we denote by B_{j_1}, \dots, B_{j_r} . We consider γ as an oriented curve from $F(w, v_i)$ to $F(w', v_{i'})$; thus we can assume that $\pi^*(\gamma)$ meets B_{j_1}, \dots, B_{j_r} in this order.

Let us define the following set of vertices in γ

$$\{w_i^1, w_i^2\} := \gamma \cap (N_{j_i} \times P_2),$$

for every $1 \leq i \leq r$, such that $d_{G_1 \times P_2}((w, v_i), w_i^1) < d_{G_1 \times P_2}((w, v_i), w_i^2)$. Note that $[w_i^2 w_{i+1}^1] \subset G'_1 \times P_2$ for every $1 \leq i < r$ and $d_{G_1 \times P_2}(w_i^2, w_{i+1}^1) \geq 1$ since $\overline{B_{j_i}} \cap \overline{B_{j_{i+1}}} = \emptyset$.

If $d_{G_1}(\pi(w_i^1), \pi(w_i^2)) = d_{G_1 \times P_2}(w_i^1, w_i^2)$ for some $1 \leq i \leq r$, then $d_{G_1 \times P_2}(w_i^1, w_i^2) \leq 2K$. Since $d_{G_1 \times P_2}(w_i^2, w_{i+1}^1) \geq 1$ for $1 \leq i < r$, we have that $d_{G_1 \times P_2}(w_i^1, w_i^2) \leq 2K d_{G_1 \times P_2}(w_i^2, w_{i+1}^1)$ in this case.

If $d_{G_1}(\pi_1(w_i^1), \pi_1(w_i^2)) < d_{G_1 \times P_2}(w_i^1, w_i^2)$ for some $1 \leq i \leq r$, then $d_{G_1}(\pi_1(w_i^1), \pi_1(w_i^2)) + d_{G_1 \times P_2}(w_i^1, w_i^2)$ is odd.

Since \mathcal{B}_J is M -regular, consider an odd cycle C with $C \cap B_{j_i} \neq \emptyset$ and $L(C) < M$, and let $b_i \in C \cap B_{j_i}$ and $[\pi_1(w_i^1)b_i], [b_i\pi_1(w_i^2)]$ geodesics in G_1 . Thus, $[\pi_1(w_i^1)b_i] \cup [b_i\pi_1(w_i^2)]$ and $[\pi_1(w_i^1)b_i] \cup C \cup [b_i\pi_1(w_i^2)]$ have different parity which means that one of them has different parity from $[\pi_1(w_i^1)\pi_1(w_i^2)]$. Then, $d_{G_1 \times P_2}(w_i^1, w_i^2) \leq L([\pi_1(w_i^1)b_i] \cup C \cup [b_i\pi_1(w_i^2)]) \leq 4K + M$. Since $d_{G_1 \times P_2}(w_i^2, w_{i+1}^1) \geq 1$ for $1 \leq i < r$, we have that $d_{G_1 \times P_2}(w_i^1, w_i^2) \leq (4K + M) d_{G_1 \times P_2}(w_i^2, w_{i+1}^1)$ in this case.

Thus, we have that $d_{G_1 \times P_2}(w_i^1, w_i^2) \leq 4K + M$ for every $1 \leq i \leq r$ and $d_{G_1 \times P_2}(w_i^1, w_i^2) \leq (4K + M) d_{G_1 \times P_2}(w_i^2, w_{i+1}^1)$ for every $1 \leq i < r$. Therefore,

$$\begin{aligned}
d_{G_1 \times P_2}((w, v_i), (w', v_{i'})) &\leq d_{G_1 \times P_2}((w, v_i), w_1^1) + \sum_{i=1}^r d_{G_1 \times P_2}(w_i^1, w_i^2) + \sum_{i=1}^{r-1} d_{G_1 \times P_2}(w_i^2, w_{i+1}^1) \\
&\quad + d_{G_1 \times P_2}(w_r^2, (w', v_{i'})) \\
&\leq d_{G_1 \times P_2}((w, v_i), w_1^1) + d_{G_1 \times P_2}(w_r^2, (w', v_{i'})) + (4K + M + 1) \sum_{i=1}^{r-1} d_{G_1 \times P_2}(w_i^2, w_{i+1}^1) \\
&\quad + d_{G_1 \times P_2}(w_r^1, w_r^2) \\
&= d_{G^*}(F(w, v_i), F(w_1^1)) + d_{G^*}(F(w_r^2), F(w', v_{i'})) + (4K + M + 1) \sum_{i=1}^{r-1} d_{G^*}(F(w_i^2), F(w_{i+1}^1)) + d_{G_1 \times P_2}(w_r^1, w_r^2) \\
&\leq (4K + M + 1) \left(d_{G^*}(F(w, v_i), F(w_1^1)) + d_{G^*}(F(w_r^2), F(w', v_{i'})) + \sum_{i=1}^{r-1} d_{G^*}(F(w_i^2), F(w_{i+1}^1)) \right) + 4K + M \\
&\leq (4K + M + 1) d_{G^*}(F(w, v_i), F(w', v_{i'})) + 4K + M.
\end{aligned}$$

We conclude that $F : V(G_1 \times P_2) \rightarrow V(G^*)$ is a quasi-isometric embedding. Thus, Lemma 2.10 provides a quasi-isometry $f : G_1 \times P_2 \rightarrow G^*$ with the required property. \square

Definition 2.33. Given a geodesic metric space X and closed connected pairwise disjoint subsets $\{\eta_j\}_{j \in J}$ of X , we consider another copy X' of X . The double DX of X is the union of X and X' obtained by identifying the corresponding points in each η_j and η'_j .

Definition 2.34. Let us consider $H > 0$, a metric space X , and subsets $Y, Z \subseteq X$. The set $V_H(Y) := \{x \in X : d(x, Y) \leq H\}$ is called the H -neighborhood of Y in X . The Hausdorff distance of Y to Z is defined by $\mathcal{H}(Y, Z) := \inf\{H > 0 : Y \subseteq V_H(Z), Z \subseteq V_H(Y)\}$.

The following results in [4] and [25] will be useful.

Theorem 2.35. [4, Theorem 3.2] Let us consider a geodesic metric space X and closed connected pairwise disjoint subsets $\{\eta_j\}_{j \in J}$ of X , such that the double DX is a geodesic metric space. Then the following conditions are equivalent:

- (1) DX is hyperbolic.
- (2) X is hyperbolic and there exists a constant c_1 such that for every $k, l \in J$ and $a \in \eta_k, b \in \eta_l$ we have $d_X(x, \cup_{j \in J} \eta_j) \leq c_1$ for every $x \in [ab] \subset X$.
- (3) X is hyperbolic and there exist constants c_2, α, β such that for every $k, l \in J$ and $a \in \eta_k, b \in \eta_l$ we have $d_X(x, \cup_{j \in J} \eta_j) \leq c_2$ for every x in some (α, β) -quasi-geodesic joining a with b in X .

Theorem 2.36. [25, p.87] For each $\delta \geq 0$, $a \geq 1$ and $b \geq 0$, there exists a constant $H = H(\delta, a, b)$ with the following property:

Let us consider a δ -hyperbolic geodesic metric space X and an (a, b) -quasigeodesic g starting in x and finishing in y . If γ is a geodesic joining x and y , then $\mathcal{H}(g, \gamma) \leq H$.

This property is known as geodesic stability. Mario Bonk proved in 1996 that geodesic stability was, in fact, equivalent to Gromov hyperbolicity (see [11]).

Theorem 2.37. Let G_1 be a graph and $B_j := B_{G_1}(w_j, K_j)$ with $w_j \in V(G_1)$ and $K_j \in \mathbb{Z}^+$, for any $j \in J$, such that $\sup_j K_j = K < \infty$, $\overline{B}_{j_1} \cap \overline{B}_{j_2} = \emptyset$ if $j_1 \neq j_2$, and every odd cycle C in G_1 satisfies $C \cap B_j \neq \emptyset$ for some $j \in J$. Suppose $\{B_j\}_{j \in J}$ is M -regular for some $M > 0$. Let G_2 be a non-trivial bounded graph without odd cycles. Then, the following statements are equivalent:

- (1) $G_1 \times G_2$ is hyperbolic.
- (2) G_1 is hyperbolic and there exists a constant c_1 , such that for every $k, l \in J$ and $w_k \in B_k$, $w_l \in B_l$ there exists a geodesic $[w_k w_l]$ in G_1 with $d_{G_1}(x, \cup_{j \in J} w_j) \leq c_1$ for every $x \in [w_k w_l]$.
- (3) G_1 is hyperbolic and there exist constants c_2, α, β , such that for every $k, l \in J$ we have $d_{G_1}(x, \cup_{j \in J} w_j) \leq c_2$ for every x in some (α, β) -quasi-geodesic joining w_k with w_l in G_1 .

Proof. Items (2) and (3) are equivalent by geodesic stability in G_1 (see Theorem 2.36).

Assume that (2) holds. By Lemma 2.29, there exists an (α, β) -quasi-isometry $f : G_1 \rightarrow G_1^*$ with $f(w_j) = w_j^*$ for every $j \in J$. Given $k, l \in J$, $f([w_k w_l])$ is an (α, β) -quasi-geodesic with endpoints w_k^* and w_l^* in G_1^* . Given $x \in f([w_k w_l])$, we have $x = f(x_0)$ with $x_0 \in [w_k w_l]$ and $d_{G_1^*}(x, \cup_{j \in J} w_j^*) \leq \alpha d_{G_1}(x_0, \cup_{j \in J} w_j) + \beta \leq \alpha c_1 + \beta$. Taking $X = G_1^*$, $DX = G^*$ and $\eta_j = w_j^*$ for every $j \in J$, Theorem 2.35 gives that G^* is hyperbolic. Now, Lemma 2.32 gives that $G_1 \times P_2$ is hyperbolic and we conclude that $G_1 \times G_2$ is hyperbolic by Lemma 2.13.

Now suppose (1) holds. By Lemma 2.13, $G_1 \times P_2$ is hyperbolic and, by Theorem 2.23, G_1 is hyperbolic. Then, Lemma 2.32 gives that G^* is hyperbolic and taking $X = G_1^*$, $DX = G^*$ and $\eta_j = w_j^*$ for every $j \in J$, by Theorem 2.35, (2) holds. \square

Theorem 2.37 and Remark 2.31 have the following consequence.

Corollary 2.38. *Let G_1 be a graph and suppose that there are a positive integer K and a vertex $w \in G_1$, such that every odd cycle in G_1 intersects the open ball $B := B_{G_1}(w, K)$. Let G_2 be a non-trivial bounded graph without odd cycles. Then, $G_1 \times G_2$ is hyperbolic if and only if G_1 is hyperbolic.*

3. BOUNDS FOR THE HYPERBOLICITY CONSTANT OF SOME DIRECT PRODUCTS

The following well-known result will be useful (see a proof, e.g., in [45, Theorem 8]).

Theorem 3.1. *In any graph G the inequality $\delta(G) \leq (\text{diam } G)/2$ holds.*

Remark 3.2. *Note that if G_1 is a bipartite graph, then $\text{diam } G_1 = \text{diam } V(G_1)$. Furthermore, if G_2 is a bipartite graph, then the product $G_1 \times G_2$ has exactly two connected components, which will be denoted by $(G_1 \times G_2)^1$ and $(G_1 \times G_2)^2$, where each one is a bipartite graph and, consequently, $\text{diam}(G_1 \times G_2)^i = \text{diam } V((G_1 \times G_2)^i)$ for $i \in \{1, 2\}$.*

Remark 3.3. *Let P_m, P_n be two path graphs with $m \geq n \geq 2$. The product $P_m \times P_n$ has exactly two connected components, which will be denoted by $(P_m \times P_n)^1$ and $(P_m \times P_n)^2$. If $u, v \in V((P_m \times P_n)^i)$ for $i \in \{1, 2\}$, then $d_{(P_m \times P_n)^i}(u, v) = \max\{d_{P_m}(\pi_1(u), \pi_1(v)), d_{P_n}(\pi_2(u), \pi_2(v))\}$ and $\text{diam}(P_m \times P_n)^i = \text{diam } V((P_m \times P_n)^i) = m - 1$.*

Furthermore, if $m_1 \leq m$ and $n_1 \leq n$ then $\delta(P_m \times P_n) \geq \delta(P_{m_1} \times P_{n_1})$.

Lemma 3.4. *Let P_m, P_n be two path graphs with $m \geq n \geq 3$, and let γ be a geodesic in $P_m \times P_n$ such that there are two different vertices u, v in γ , with $\pi_1(u) = \pi_1(v)$. Then, $L(\gamma) \leq n - 1$.*

Proof. Let $\gamma := [xy]$, and let $V(P_m) = \{v_1, \dots, v_m\}$, $V(P_n) = \{w_1, \dots, w_n\}$ be the sets of vertices in P_m, P_n , respectively, such that $[v_j, v_{j+1}] \in E(P_m)$ and $[w_i, w_{i+1}] \in E(P_n)$ for $1 \leq j < m, 1 \leq i < n$. Seeking for a contradiction, assume that $L(\gamma) > n - 1$. Notice that if $[uv]$ denotes the geodesic contained in γ joining u and v , then π_2 restricted to $[uv]$ is injective. Consider two vertices $u', v' \in \gamma$ such that $[uv] \subseteq [u'v'] \subseteq \gamma$, π_2 is injective in $[u'v']$ and $\pi_2(u') = w_{i_1}$, $\pi_2(v') = w_{i_2}$ with $i_2 - i_1$ maximal under these conditions. Since $L(\gamma) > n - 1 \geq i_2 - i_1$, either there is an edge $[v', w]$ in $G_1 \times G_2$ such that $[v', w] \cap (\gamma \setminus [u'v']) \neq \emptyset$ or there is an edge $[u', w']$ in $G_1 \times G_2$ such that $[u', w'] \cap (\gamma \setminus [u'v']) \neq \emptyset$. Also, since $L(\gamma) > n - 1$, notice that π_2 is not injective in γ . Moreover, since $i_2 - i_1$ is maximal, if $\pi_2(w) = w_{i_2+1}$, then $w \notin \gamma$, and since $L(\gamma) > n - 1$, $u' \notin \{x, y\}$ and $\pi_2(w') = w_{i_1+1}$. Thus, either $\pi_2(w) = w_{i_2-1}$ or $\pi_2(w') = w_{i_1+1}$.

Hence, let us assume that there is an edge $[v', w]$ in $G_1 \times G_2$ such that $[v', w] \cap (\gamma \setminus [u'v']) \neq \emptyset$ with $\pi_2(w) = w_{i_2-1}$ (otherwise, if there is an edge $[u', w']$ in $G_1 \times G_2$ such that $[u', w'] \cap (\gamma \setminus [u'v']) \neq \emptyset$ with $\pi_2(w') = w_{i_1+1}$, the proof is similar).

Suppose $\pi_1(v') = v_j$. Let v'' be the vertex in $[u'v']$ such that $\pi_2(v'') = w_{i_2-1}$. Then, by construction of $G_1 \times G_2$, since $v'' \neq w$, it follows that $\{\pi_1(v''), \pi_1(w)\} = \{v_{j-1}, v_{j+1}\}$. Therefore, in particular, $1 < j < m$.

Assume that $v'' = (v_{j-1}, w_{i_2-1})$ (if $v'' = (v_{j+1}, w_{i_2-1})$, then the argument is similar). Therefore, $w = (v_{j+1}, w_{i_2-1})$.

Consider the geodesic

$$\sigma = [(v_{j+1}, w_{i_2-1}), (v_j, w_{i_2-2})] \cup [(v_j, w_{i_2-2}), (v_{j-1}, w_{i_2-3})] \cup [(v_{j-1}, w_{i_2-3}), (v_{j-2}, w_{i_2-4})] \cup \dots$$

Since $\pi_1(u) = \pi_1(v)$, there is a vertex ξ of $V(P_m \times P_n)$ in $[u'v'] \cap \sigma$. Let $s \in [v', w] \cap \gamma$ with $s \neq v'$. Let σ_0 be the geodesic contained in σ joining ξ and w . Let γ_0 be the geodesic contained in γ joining ξ and s . Hence, $L(\sigma_0 \cup [ws]) < L(\sigma_0) + 1 < L(\gamma_0)$ leading to contradiction. \square

Theorem 3.5. *Let P_m, P_n be two path graphs with $m \geq n \geq 2$. If $n = 2$, then $\delta(P_m \times P_2) = 0$. If $n \geq 3$, then*

$$\min \left\{ \frac{m}{2}, n-1 \right\} - 1 \leq \delta(P_m \times P_n) \leq \min \left\{ \frac{m}{2}, n \right\} - \frac{1}{2}.$$

Furthermore, if $m \leq 2n-3$ and m is odd, then $\delta(P_m \times P_n) = (m-1)/2$.

Proof. If $m \geq 2$, then $P_m \times P_2$ has two connected components isomorphic to P_m , and $\delta(P_m \times P_2) = 0$.

Assume that $n \geq 3$. By symmetry, it suffices to prove the inequalities for $\delta((P_m \times P_n)^1)$. Hence, Theorem 3.1 and Remark 3.3 give $\delta((P_m \times P_n)^1) \leq \frac{m-1}{2}$. By Theorem 2.21, there exists a geodesic triangle $T = \{x, y, z\} \in \mathbb{T}_1(P_m \times P_n)$ with $p \in \gamma_1 := [xy], \gamma_2 := [xz], \gamma_3 := [yz]$, and $\delta((P_m \times P_n)^1) = \delta(T) = d_{(P_m \times P_n)^1}(p, \gamma_2 \cup \gamma_3)$. Let $u \in V(\gamma_1)$ such that $d_{(P_m \times P_n)^1}(p, u) \leq 1/2$.

In order to prove $\delta((P_m \times P_n)^1) \leq n-1/2$, we consider two cases.

Assume first that there is at least a vertex $v \in V((P_m \times P_n)^1) \cap T \setminus \{u\}$ such that $\pi_1(u) = \pi_1(v)$. If $v \notin \gamma_1$, then $v \in \gamma_2 \cup \gamma_3$ and

$$\delta(T) = d_{(P_m \times P_n)^1}(p, \gamma_2 \cup \gamma_3) \leq 1/2 + d_{(P_m \times P_n)^1}(u, v) \leq n-1/2.$$

If $v \in \gamma_1$, then $L(\gamma_1) \leq n-1$ by Lemma 3.4, and

$$\delta(T) = d_{(P_m \times P_n)^1}(p, \gamma_2 \cup \gamma_3) \leq d_{(P_m \times P_n)^1}(p, \{x, y\}) \leq (n-1)/2 < n-1/2.$$

Assume now that there is not a vertex $v \in V((P_m \times P_n)^1) \cap T \setminus \{u\}$ such that $\pi_1(u) = \pi_1(v)$. Then, there exist two different vertices v_1, v_2 in $T \setminus \{u\}$ such that $d_{(P_m \times P_n)^1}(u, v_1) = d_{(P_m \times P_n)^1}(u, v_2) = 1$, and $\pi_1(v_1) = \pi_1(v_2)$. If v_1 or v_2 belongs to $\gamma_2 \cup \gamma_3$, then $\delta(T) = d_{(P_m \times P_n)^1}(p, \gamma_2 \cup \gamma_3) \leq 3/2 \leq n-1/2$. Otherwise, $v_1, v_2 \in \gamma_1 \setminus \{u\}$. Lemma 3.4 gives $L(\gamma_1) \leq n-1$, and we have that

$$\delta(T) = d_{(P_m \times P_n)^1}(p, \gamma_2 \cup \gamma_3) \leq d_{(P_m \times P_n)^1}(p, \{x, y\}) \leq (n-1)/2 < n-1/2.$$

In order to prove the lower bound, denote the vertices of P_m and P_n by $V(P_m) = \{w_1, w_2, w_3, \dots, w_m\}$ and $V(P_n) = \{v_1, v_2, v_3, \dots, v_n\}$, with $[w_i, w_{i+1}] \in E(P_m)$ for $1 \leq i < m$ and $[v_i, v_{i+1}] \in E(P_n)$ for $1 \leq i < n$.

Let $(P_m \times P_n)^1$ be the connected component of $P_m \times P_n$ containing (w_1, v_{n-1}) .

Assume first that $m \geq 2n-3$. Consider the following curves in $(P_m \times P_n)^1$:

$$\begin{aligned} \gamma_1 &:= [(w_1, v_{n-1}), (w_2, v_n)] \cup [(w_2, v_n), (w_3, v_{n-1})] \cup [(w_3, v_{n-1}), (w_4, v_n)] \cup \dots \cup [(w_{2n-4}, v_n), (w_{2n-3}, v_{n-1})], \\ \gamma_2 &:= [(w_1, v_{n-1}), (w_2, v_{n-2})] \cup [(w_2, v_{n-2}), (w_3, v_{n-3})] \cup \dots \cup [(w_{n-2}, v_2), (w_{n-1}, v_1)] \cup [(w_{n-1}, v_1), (w_n, v_2)] \\ &\quad \cup \dots \cup [(w_{2n-4}, v_{n-2}), (w_{2n-3}, v_{n-1})]. \end{aligned}$$

Corollary 2.5 gives that γ_1, γ_2 are geodesics. If B is the geodesic bigon $B = \{\gamma_1, \gamma_2\}$, then Remark 3.3 gives that

$$\delta(P_m \times P_n) \geq \delta(B) \geq d_{(P_m \times P_n)^1}((w_{n-1}, v_1), \gamma_1) = n-2.$$

If m is odd with $m \leq 2n - 3$, then $n - (m + 1)/2 \geq 1$ and we can consider the curves in $(P_m \times P_n)^1$:

$$\begin{aligned}\gamma_1 &:= [(w_1, v_{n-1}), (w_2, v_n)] \cup [(w_2, v_n), (w_3, v_{n-1})] \cup [(w_3, v_{n-1}), (w_4, v_n)] \cup \cdots \cup [(w_{m-1}, v_n), (w_m, v_{n-1})], \\ \gamma_2 &:= [(w_1, v_{n-1}), (w_2, v_{n-2})] \cup [(w_2, v_{n-2}), (w_3, v_{n-3})] \cup \cdots \cup [(w_{(m+1)/2-1}, v_{n-(m+1)/2+1}), (w_{(m+1)/2}, v_{n-(m+1)/2})] \\ &\quad \cup [(w_{(m+1)/2}, v_{n-(m+1)/2}), (w_{(m+1)/2+1}, v_{n-(m+1)/2+1})] \cup \cdots \cup [(w_{m-1}, v_{n-2}), (w_m, v_{n-1})].\end{aligned}$$

Corollary 2.5 gives that γ_1, γ_2 are geodesics. If $B = \{\gamma_1, \gamma_2\}$, then Remark 3.3 gives that

$$\delta(P_m \times P_n) \geq \delta(B) \geq d_{(P_m \times P_n)^1}((w_{(m+1)/2}, v_{n-(m+1)/2}), \gamma_1) = (m - 1)/2.$$

By Remark 3.3, if m is even with $m - 1 \leq 2n - 3$, then we have that

$$\delta(P_m \times P_n) \geq \delta(P_{m-1} \times P_n) \geq (m - 2)/2.$$

Hence,

$$\delta(P_m \times P_n) \geq \begin{cases} n - 2, & \text{if } m \geq 2n - 3 \\ (m - 2)/2, & \text{if } m \leq 2n - 2 \end{cases} = \min \left\{ n - 2, \frac{m - 2}{2} \right\} = \min \left\{ \frac{m}{2}, n - 1 \right\} - 1.$$

Furthermore, if $m \leq 2n - 3$ and m is odd, then we have proved $(m - 1)/2 \leq \delta(P_m \times P_n) \leq (m - 1)/2$. \square

Theorem 3.6. *If G_1 and G_2 are bipartite graphs with $k_1 := \text{diam } V(G_1)$ and $k_2 := \text{diam } V(G_2)$ such that $k_1 \geq k_2 \geq 1$, then*

$$\max \left\{ \min \left\{ \frac{k_1 - 1}{2}, k_2 - 1 \right\}, \delta(G_1), \delta(G_2) \right\} \leq \delta(G_1 \times G_2) \leq \frac{k_1}{2}.$$

Furthermore, if $k_1 \leq 2k_2 - 2$ and k_1 is even, then $\delta(G_1 \times G_2) = k_1/2$.

Proof. Corollary 2.5, Theorem 3.1 and Remark 3.2 give us the upper bound.

In order to prove the lower bound, we can see that there exist two path graphs P_{k_1+1}, P_{k_2+1} which are isometric subgraphs of G_1 and G_2 , respectively. It is easy to check that $P_{k_1+1} \times P_{k_2+1}$ is an isometric subgraph of $G_1 \times G_2$. By Lemma 2.14 and Theorem 3.5, we have

$$\min \left\{ \frac{k_1 - 1}{2}, k_2 - 1 \right\} \leq \delta(P_{k_1+1} \times P_{k_2+1}) \leq \delta(G_1 \times G_2).$$

Using a similar argument as above, we have $\delta(P_2 \times G_2) \leq \delta(G_1 \times G_2)$ and $\delta(G_1 \times P_2) \leq \delta(G_1 \times G_2)$. Thus, since $(G_1 \times P_2)^i \simeq G_1$ and $(P_2 \times G_2)^i \simeq G_2$ for $i \in \{1, 2\}$, we obtain the first statement.

Furthermore, if $k_1 + 1 \leq 2(k_2 + 1) - 3$ and $k_1 + 1$ is odd, then Theorem 3.5 gives $\delta(P_{k_1+1} \times P_{k_2+1}) = k_1/2$, and we conclude $\delta(G_1 \times G_2) = k_1/2$. \square

The following result deals just with odd cycles since otherwise we can apply Theorem 3.6.

Theorem 3.7. *For every odd number $m \geq 3$ and every $n \geq 2$,*

$$\delta(C_m \times P_n) = \begin{cases} m/2, & \text{if } n - 1 \leq m, \\ (n - 1)/2, & \text{if } m < n - 1 < 2m, \\ m - 1/2, & \text{if } n - 1 \geq 2m. \end{cases}$$

Proof. Let $V(C_m) = \{w_1, \dots, w_m\}$ and $V(P_n) = \{v_1, \dots, v_n\}$ be the sets of vertices in C_m and P_n , respectively, such that $[w_1, w_m], [w_j, w_{j+1}] \in E(C_m)$ and $[v_i, v_{i+1}] \in E(P_n)$ for $j \in \{1, \dots, m-1\}$, $i \in \{1, \dots, n-1\}$. Note that for $1 \leq j, r \leq m$ and $1 \leq i, s \leq n$, we have $d_{C_m \times P_n}((w_j, v_i), (w_r, v_s)) = \max\{|i - s|, |j - r|\}$, if $|i - s| \equiv |j - r| \pmod{2}$, or $d_{C_m \times P_n}((w_j, v_i), (w_r, v_s)) = \max\{|i - s|, m - |j - r|\}$, if $|i - s| \not\equiv |j - r| \pmod{2}$. Besides, we have $\text{diam}(C_m \times P_n) = \text{diam}(V(C_m \times P_n))$, i.e., $\text{diam}(C_m \times P_n) = m$ if $n - 1 \leq m$, and $\text{diam}(C_m \times P_n) = n - 1$ if $n - 1 > m$. Thus, by Theorem 3.1 we have

$$\delta(C_m \times P_n) \leq \begin{cases} m/2, & \text{if } n - 1 \leq m, \\ (n - 1)/2, & \text{if } n - 1 > m. \end{cases}$$

Assume first that $n - 1 \leq m$. Note that $C_m \times P_2 \simeq C_{2m}$ and $C_m \times P_{n'}$ is an isometric subgraph of $C_m \times P_n$, if $n' \leq n$. By Lemma 2.14, we have $\delta(C_m \times P_n) \geq \delta(C_{2m}) = m/2$, and we obtain the result in this case.

Assume now that $n - 1 > m$. Consider the geodesic triangle T in $C_m \times P_n$ defined by the following geodesics

$$\begin{aligned} \gamma_1 &:= [(w_1, v_n), (w_2, v_{n-1})] \cup [(w_2, v_{n-1}), (w_3, v_n)] \cup [(w_3, v_n), (w_4, v_{n-1})] \cup \dots \cup [(w_{m-1}, v_{n-1}), (w_m, v_n)], \\ \gamma_2 &:= [(w_{(m+1)/2}, v_1), (w_{(m-1)/2}, v_2)] \cup [(w_{(m-1)/2}, v_2), (w_{(m-3)/2}, v_3)] \cup \dots \cup [(w_2, v_{(m-1)/2}), (w_1, v_{(m+1)/2})] \cup \\ &\quad [(w_1, v_{(m+1)/2}), (w_m, v_{(m+3)/2})] \cup [(w_m, v_{(m+3)/2}), (w_1, v_{(m+5)/2})] \cup [(w_1, v_{(m+5)/2}), (w_m, v_{(m+7)/2})] \cup \dots, \\ \gamma_3 &:= [(w_{(m+1)/2}, v_1), (w_{(m+3)/2}, v_2)] \cup [(w_{(m+3)/2}, v_2), (w_{(m+5)/2}, v_3)] \cup \dots \cup [(w_{m-1}, v_{(m-1)/2}), (w_m, v_{(m+1)/2})] \cup \\ &\quad [(w_m, v_{(m+1)/2}), (w_1, v_{(m+3)/2})] \cup [(w_1, v_{(m+3)/2}), (w_m, v_{(m+5)/2})] \cup [(w_m, v_{(m+5)/2}), (w_1, v_{(m+7)/2})] \cup \dots, \end{aligned}$$

where (w_1, v_n) (respectively, (w_m, v_n)) is an endpoint of either γ_2 or γ_3 , depending of the parity of n . Since T is a geodesic triangle in $C_m \times P_n$, we have $\delta(C_m \times P_n) \geq \delta(T)$. If $n - 1 < 2m$ and M is the midpoint of the geodesic γ_3 , then $\delta(C_m \times P_n) \geq \delta(T) = d_{C_m \times P_n}(M, \gamma_1 \cup \gamma_2) = L(\gamma_3)/2 = (n - 1)/2$. Therefore, the result for $m < n - 1 < 2m$ follows.

Finally, assume that $n - 1 \geq 2m$. Let us consider $N \in \gamma_3$ such that $d_{C_m \times P_n}(N, (w_{(m+1)/2}, v_1)) = m - 1/2$. Thus, $\delta(C_m \times P_n) \geq \delta(T) \geq d_{C_m \times P_n}(N, \gamma_1 \cup \gamma_2) = d_{C_m \times P_n}(N, (w_{(m+1)/2}, v_1)) = m - 1/2$. In order to finish the proof, it suffices to prove that $\delta(C_m \times P_n) \leq m - 1/2$. Seeking for a contradiction, assume that $\delta(C_m \times P_n) > m - 1/2$. By Theorems 2.20 and 2.21, there is a geodesic triangle $\Delta = \{x, y, z\} \in \mathbb{T}_1(C_m \times P_n)$ and $p \in [xy]$ with $d_{C_m \times P_n}(p, [yz] \cup [zx]) = \delta(C_m \times P_n) \geq m - 1/4$. Then, $L([xy]) = d_{C_m \times P_n}(x, p) + d_{C_m \times P_n}(p, y) \geq 2m - 1/2$. Let V_x (respectively, V_y) be the closest vertex to x (respectively, y) in $[xy]$, and consider a vertex V_p in $[xy]$ such that $d_{C_m \times P_n}(p, V(C_m \times P_n)) = d_{C_m \times P_n}(p, V_p)$. Note that $d_{C_m \times P_n}(p, [yz] \cup [zx]) \geq m - 1/4$ implies that $d_{C_m \times P_n}(p, V_p) \leq 1/2$. Since $x, y, z \in J(C_m \times P_n)$, we have $d_{C_m \times P_n}(V_x, V_y) \geq 2m - 1 > m$ and, consequently, $\pi_2([xy])$ is a geodesic in P_n . Since $\pi_2([yz] \cup [zx])$ is a path in P_n joining $\pi_2(x)$ and $\pi_2(y)$, there exists a vertex $(u, v) \in [xz] \cup [zy]$ such that $\pi_2(V_p) = v$ and $u \neq \pi_1(V_p)$. Therefore, $d_{C_m \times P_n}(V_p, (u, v)) \leq m - 1$ and, consequently, $d_{C_m \times P_n}(p, [xz] \cup [zy]) \leq d_{C_m \times P_n}(p, V_p) + d_{C_m \times P_n}(V_p, [xz] \cup [zy]) \leq 1/2 + m - 1$, leading to contradiction. \square

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